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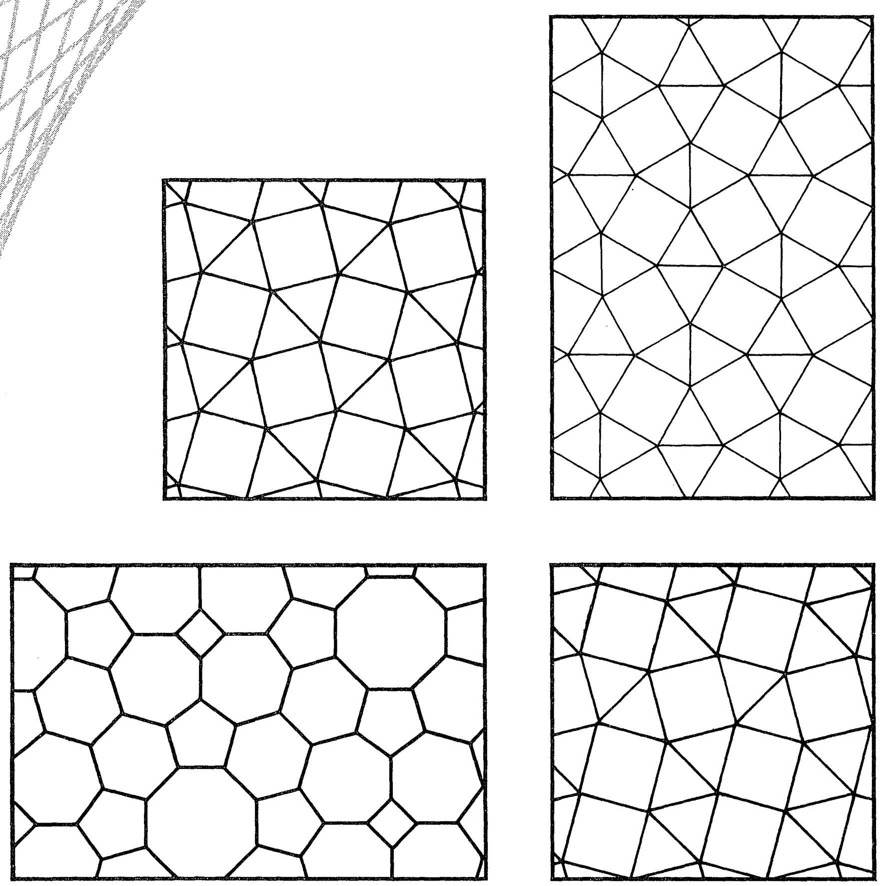

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MATHEMATICS MAGAZINE

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COVER: Despite appearances, one of these four tilings is not a tiling by regular polygons. The subject of regular tilings is more subtle than the eye; a detailed treatment is given in the article beginning on p. 227.

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ABOUT OUR AUTHORS

Branko Grünbaum and G. C. Shephard ("Tilings by Regular Polygons") have for many years cooperated in research on convexity and other geometric topics. At a meeting in Durham (England) in the summer of 1975 they "discovered" tilings — or, rather, that in this attractive part of geometry there are many facts that deserve to be more widely known, and many more that first have to be established. Ever since then they have been pursuing various ramifications of this topic, which is relevant also to crystallography, art and other disciplines. They have obtained a number of new results on tilings, some of which are being published in several research papers; they plan to present a detailed account of all this material in a planned volume "Tilings and Patterns". The general awakening of interest in the topic is attested to by the three-day Special Session on "Tilings, Patterns and Symmetries" held at the Summer Meeting of the AMS in August 1977.

Lawrence D. Stone ("Search Theory: A Mathematical Theory for Finding Lost Objects") received his B.S. in Mathematics from Antioch College (1964) and his M.S. (1966) and Ph.D. (1967) in mathematics from Purdue University, and then joined Daniel H. Wagner, Associates. In 1968 he participated in the successful search for the remains of the nuclear submarine Scorpion. As a result of this experience, he coauthored a manual for operations analysis of ocean-bottom searches. In 1974 he spent seven weeks at the Suez Canal providing analysis assistance for the operation to clear unexploded ordnance from the canal. He has also assisted in the development of the U.S. Coast Guard's Computer Assisted Search System (CASP), and his book, *Theory of Optimal Search,* received the Operations Research Society of America's Lanchester prize in 1975.

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Tilings by Regular Polygons

Patterns in the plane from Kepler to the present, including recent results and unsolved problems.

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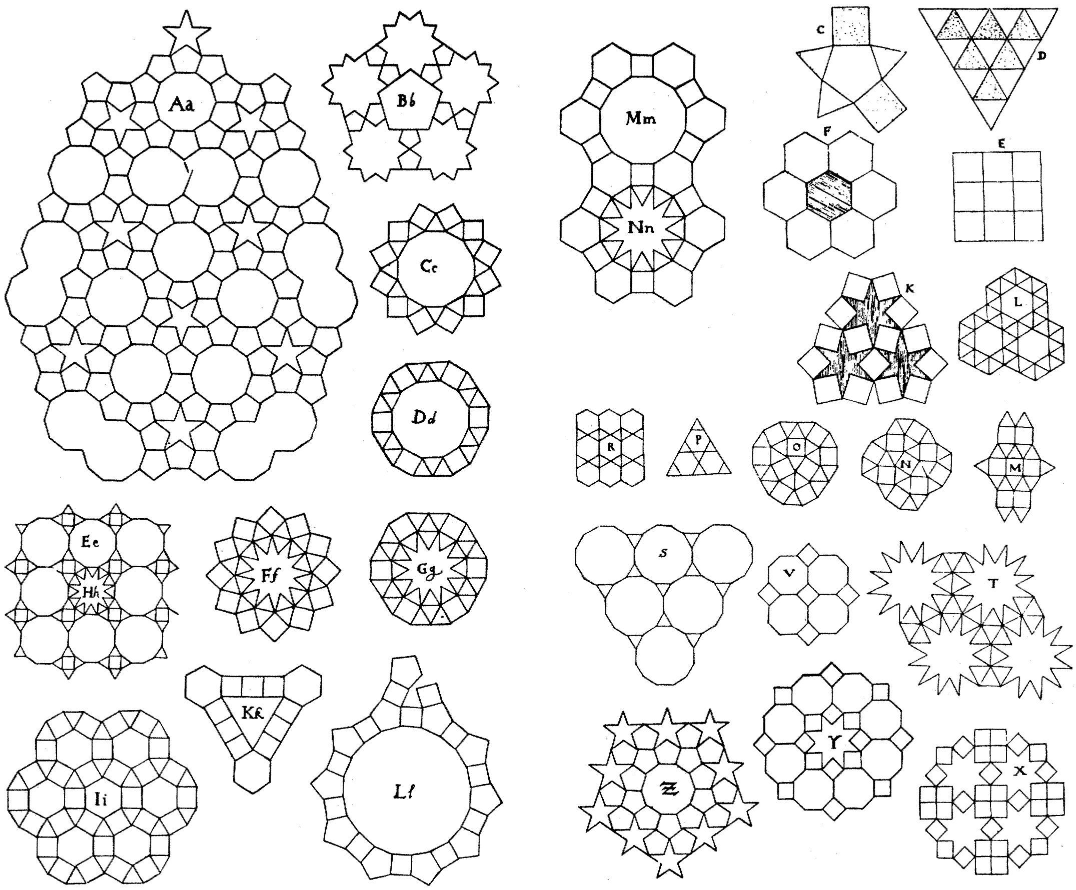
A **tiling** of the plane is a family of sets — called **tiles** — that cover the plane without gaps or overlaps. ("Without overlaps" means that the intersection of any two of the sets has measure (area) zero.) Tilings are also known as **tessellations, pavings,** or **mosaics;** they have appeared in human activities since prehistoric times. Their mathematical theory is mostly elementary, but nevertheless it contains a rich supply of interesting and sometimes surprising facts as well as many challenging problems at various levels. The same is true for the special class of tilings that will be discussed here — more or less regular tilings by regular polygons. These types were chosen because they are accessible without any need for lengthy introductions, and also because they were the first to be the subject of mathematical research. The pioneering investigation was done by Johannes Kepler, more than three and a half centuries ago. Additional historical data will be given later (in Section 6) but as an introduction we reproduce in FIGURE 1 certain drawings from Kepler [1619]. We shall see that these drawings contain (at least in embryonic form) many aspects of tilings by regular polygons which even at present are not completely developed.

As is the case with many other notions, the concept of "more or less regular" tilings by regular polygons developed through the centuries in response to the interests of various investigators; it is still changing, and no single point of view can claim absolute superiority over all others. Our presentation reflects our preferences, although many other definitions and directions are possible; some of these will be briefly indicated in Sections 4, 5 and 7. For most of our assertions we provide only hints which we hope will be sufficient for interested readers to construct complete proofs.

Initially we shall use only regular convex polygons as tiles: if such a polygon has n edges (or sides) we shall call it an ***n*-gon,** and use for it the symbol $\{n\}$. Thus $\{3\}$ denotes an equilateral triangle, while $\{4\}$, $\{5\}$, $\{6\}$ denote a square, a (regular) pentagon, and a (regular) hexagon, respectively. All the polygons are understood to be closed sets, that is, to include their edges and vertices.



Various more or less regular tilings of the plane by regular polygons, reproduced from J. Kepler's book "Harmonices Mundi", published in 1619.

FIGURE 1

Except in Section 4 we shall restrict attention to tilings that are **edge-to-edge;** by this we mean that as far as the mutual relation of any two tiles is concerned there are just three possibilities:

(i) they are disjoint (have no point in common);

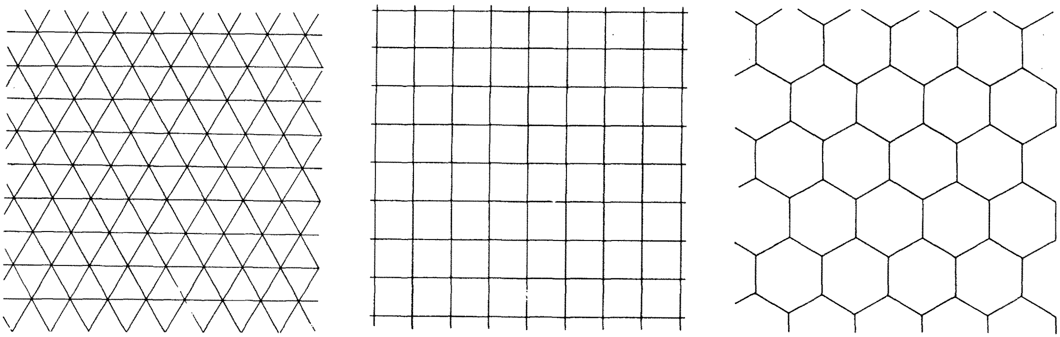
(ii) they have precisely one common point which is a vertex of each of the polygons; or

(iii) they share a segment that is an edge of each of the two polygons.

Hence a point of the plane that is a vertex of one of the polygons in an edge-to-edge tiling is also a vertex of every other polygon to which it belongs; we shall say that it is a **vertex of the tiling.** Similarly, each edge of one of the polygons is an edge of precisely one other polygon and we call it an **edge of the tiling.**

1. Regular and uniform tilings

The question about the possibilities of tiling the plane by (congruent) copies of a single regular polygon has the following simple and rather obvious answer, the origin of which is lost in antiquity. *The only possible edge-to-edge tilings of the plane by mutually congruent regular convex polygons are the three regular tilings by equilateral triangles, by squares, or by regular hexagons.* A portion of each of these three tilings is illustrated in FIGURE 2.



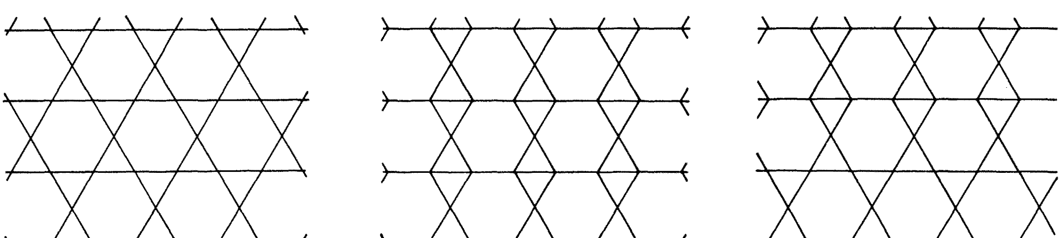
(3^6) (4^4) (6^3)

The three regular tilings of the plane.

FIGURE 2

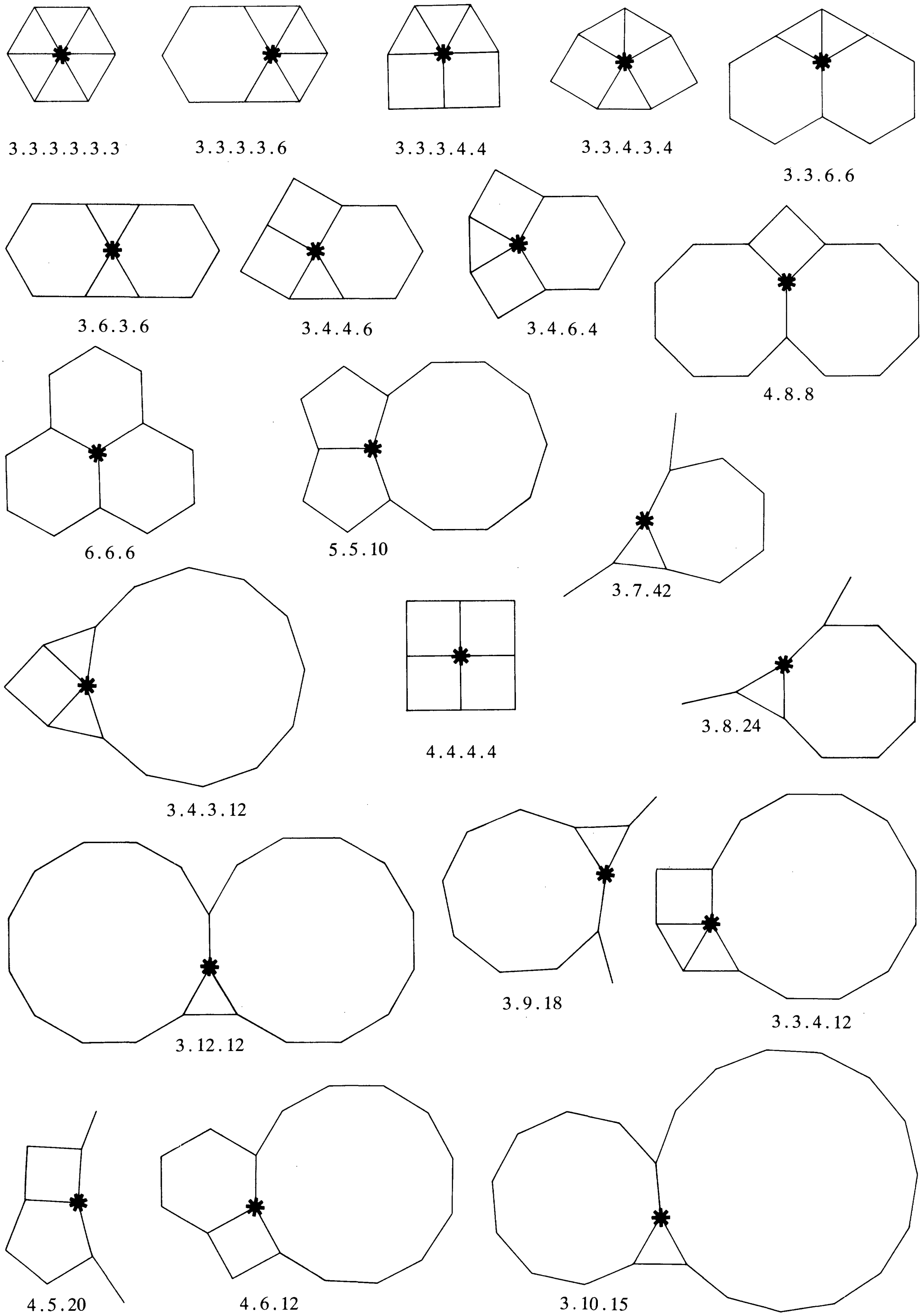
If we inquire about the possibility of edge-to-edge tilings of the plane that use as tiles regular polygons of several kinds, then the situation immediately becomes much more interesting. The angle at each vertex of $\{n\}$ is $(n-2)\pi/n$ so it is easy to check by simple arithmetic that only 17 choices of polygons can be fitted around a single vertex so as to cover a neighborhood of the vertex without gaps or overlaps. We call each such choice the **species** of the vertex, and list in TABLE 1 the 17 possible species. In four of the species there are two distinct ways in which the polygons in question may be arranged around a vertex; the mere reversal of cyclic order is not counted as distinct. Hence there are 21 possible **types** of vertices; they too are listed in TABLE 1 and also illustrated in FIGURE 3. We denote the type of a vertex around which there are, in cyclic order, an a-gon $\{a\}$, a b-gon $\{b\}$, a c-gon $\{c\}$, etc., by $a.b.c.\ldots..$ Thus the three regular tilings have vertices of types $3.3.3.3.3.3$, $4.4.4.4$, and $6.6.6$. For brevity we shall write these symbols as 3^6, 4^4 and 6^3, and we shall use similar abbreviations in other cases. In order to obtain a unique symbol for each type of vertex we shall always choose that which is lexicographically first among all possible expressions.

Contrary to frequently made assertions (see Section 6), if we require of an edge-to-edge tiling only that it be composed of regular polygons and that all its vertices be of the same species, then there are infinitely many distinct types of tilings. For example (see FIGURE 4), if at each vertex there are two triangles and two hexagons, it is possible to place each "horizontal" strip in two non-equivalent



By sliding horizontal strips independently of each other, an uncountable infinity of distinct tilings may be obtained, all vertices of which are of species 5.

FIGURE 4



The 21 possible types of vertices.

FIGURE 3

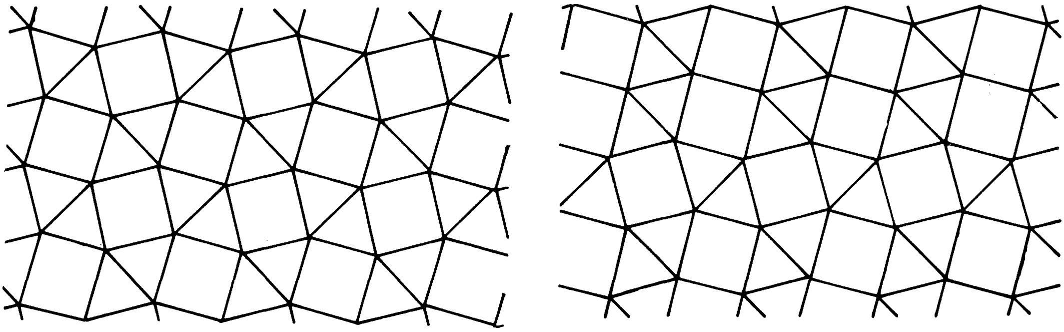
positions. Since there are infinitely many such strips, there will be uncountably many distinct tilings. The situation in FIGURE 5 in which 3 triangles meet 2 squares at each vertex is similar. In FIGURE 6 we allow three kinds of polygons; this permits each "disc" of an infinite family to be put in two positions, again leading to uncountably many tilings, with each vertex of species 6.

In view of the above remarks it is reasonable to restrict attention to tilings in which only a single **type** of vertex is allowed. If that type is $a.b.c.\ldots$, we shall denote the tiling by $(a.b.c.\ldots)$, using superscripts to shorten the expression when possible. This restriction indeed changes the situation completely and we have the following result: *There exist precisely* 11 *distinct types of edge-to-edge tilings by regular polygons such that all vertices of the tiling are of the same type.* These 11 types of

Species number	$n=3$	4	5	6	7	8	9	10	12	15	18	20	24	42	Type of vertex	Type of tiling
1	6														3.3.3.3.3.3	*A*
2	4			1											3.3.3.3.6	*A*
3	3	2													3.3.3.4.4	*A*
															3.3.4.3.4	*A*
4	2	1							1						3.3.4.12	
															3.4.3.12	
5	2			2											3.3.6.6	
															3.6.3.6	*A*
6	1	2		1											3.4.4.6	
															3.4.6.4	*A*
7	1				1									1	3.7.42	
8	1					1							1		3.8.24	
9	1						1				1				3.9.18	
10	1							1		1					3.10.15	
11	1								2						3.12.12	*A*
12		4													4.4.4.4	*A*
13		1	1									1			4.5.20	
14		1		1					1						4.6.12	*A*
15		1				2									4.8.8	*A*
16			2					1							5.5.10	
17				3											6.6.6	*A*

POSSIBLE SPECIES AND TYPES of vertices for edge-to-edge tilings by regular polygons. Entries in the table indicate the number of n-gons that meet at a vertex. Types that lead to Archimedean tilings are labelled with an "A" in the final column.

TABLE 1

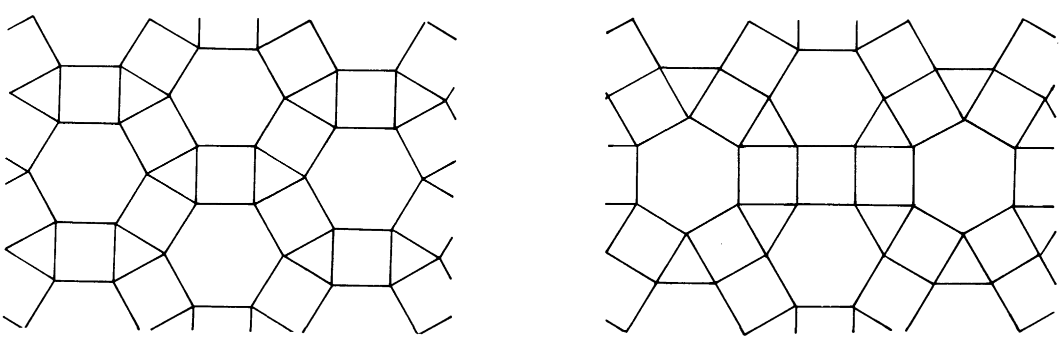


Infinitely many distinct tilings that have only vertices of species 3 may be obtained by changing the relative positions of horizontal zigzag strips in the tiling at the left.

FIGURE 5

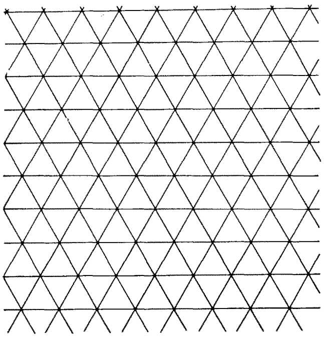
tilings, illustrated in FIGURE 7, are usually called **Archimedean** tilings (although some authors call them homogeneous, or semiregular, or uniform). They clearly include the three types of regular tilings.

Two not entirely trivial steps are required in order to prove that there are precisely 11 types of Archimedean tilings. In the first place, it must be shown that for ten of the 21 types of vertices listed in TABLE 1 it is not possible to extend a tiling from the neighborhood of a starting vertex to an Archimedean tiling of the whole plane. In fact, in each case one has to go only around one of the n-gons with odd n to show the impossibility. (For each of the six species numbered 7, 8, 9, 10 13 and 16 there is no edge-to-edge tiling of the plane by regular polygons that includes even a single vertex of the species.) In the second place, it must be established that the remaining 11 types of vertices do actually lead to Archimedean tilings. This may be deemed obvious and trivial in view of FIGURE 7, but it is just this "obviousness" that is dangerous. In FIGURE 8, adapted from a children's coloring book, we show a tiling that appears to consist of regular n-gons with $n = 4, 5, 6, 7, 8$. Actually, this visual "proof" is a fraud, since it is easy to check that the polygons in such a tiling cannot be regular. Thus there is a real need to show that the 11 Archimedean tilings do exist. It is easy to give direct proofs of existence for (4^4) and for (3^6) by considering two or three suitable families of equidistant parallel lines. The existence of the other Archimedean tilings can be deduced (with just a little thought) from these two.

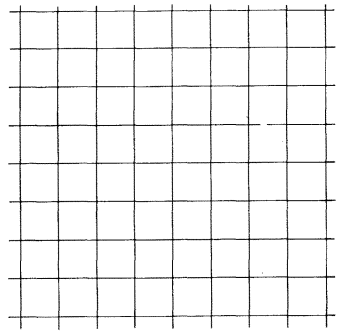


By turning "discs" in the tiling at the left infinitely many different tilings with all vertices of species 6 may be obtained.

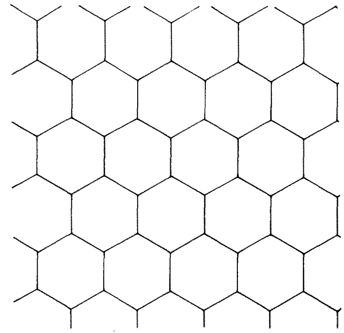
FIGURE 6



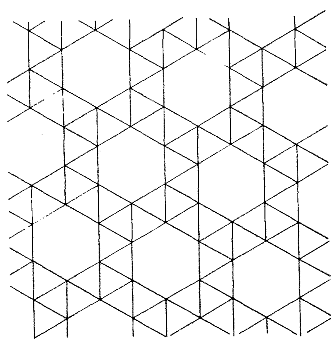
(3^6)



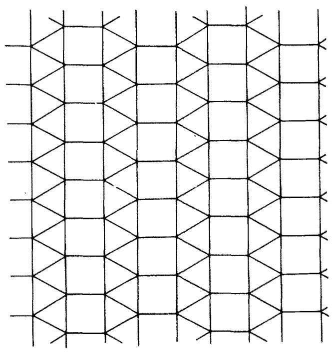
(4^4)



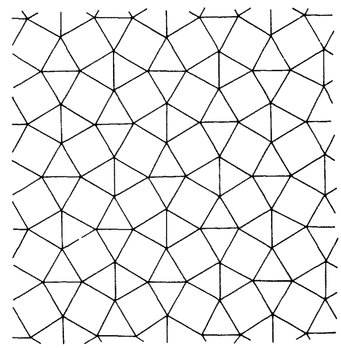
(6^3)



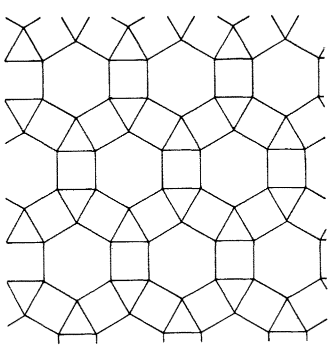
$(3^4.6)$



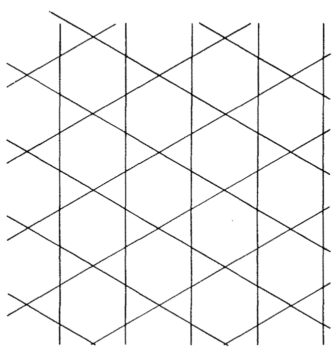
$(3^3.4^2)$



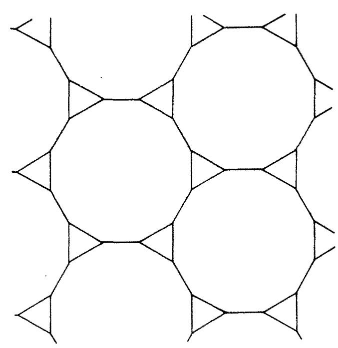
$(3^2.4.3.4)$



(3.4.6.4)



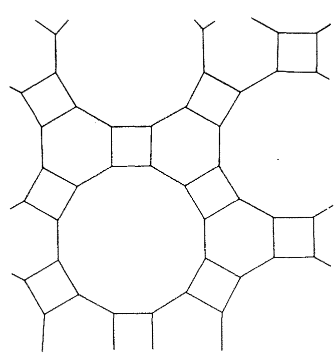
(3.6.3.6)



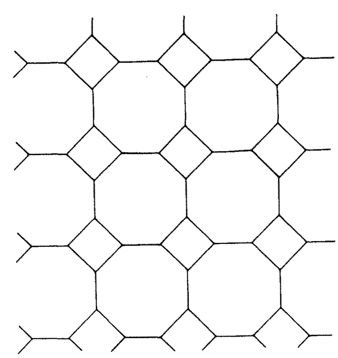
(3.12^2)

The 11 distinct types of Archimedean tilings of the plane. The tiling of type $(3^4.6)$ exists in two mirror-symmetric (enantiomorphic) forms.

FIGURE 7



(4.6.12)



(4.8^2)

It should be noted that *a priori* it is not obvious that the limitation to a single type of vertex should lead to a single type of tiling. It happens to turn out that way, but just barely so: it is only because we find it convenient not to distinguish between tilings that are congruent but not directly congruent. Indeed, the tilings of type $(3^4.6)$ are of two mirror-symmetric (enantiomorphic) forms that are counted as distinct by some authors.

Another accidental but very important feature of the Archimedean tilings is the fact that each is **vertex-transitive.** By this we mean that all vertices are equivalent under the symmetries of the tiling. Put more simply, for each pair of vertices A and B it is possible to find a motion of the plane, or a motion combined with a reflection in a line, that carries the tiling onto itself and maps A onto B. A verification of the vertex-transitivity of the Archimedean tilings is a very useful exercise. A psychologically very convincing (although logically not completely conclusive) verification of the transitivity may be obtained by tracing the tiling on a transparent sheet that may be moved over the original, and turned over. (Note that a tiling may be vertex-transitive even if its tiles are not regular polygons. Some examples of such tilings will be found in FIGURES 14 and 16.) In view of the transitivity of Archimedean tilings we shall from now on also call them **uniform** tilings. The distinction between the two words is that "Archimedean" refers only to the fact that the immediate neighborhoods of any two vertices "look the same", while the term "uniform" implies the much stronger property of equivalence of vertices under symmetries of the whole tiling.

Returning to the question of tiling with a single species of vertex we mention without proof that non-uniform tilings are possible only in case of species 3, 5 and 6. In the last two of those cases *all* tilings can be obtained from the uniform ones, $(3.6.3.6)$ and $(3.4.6.4)$, by the method explained above. However, in case of species 3 there are other possibilities as well and a complete description of all such tilings is still not known.

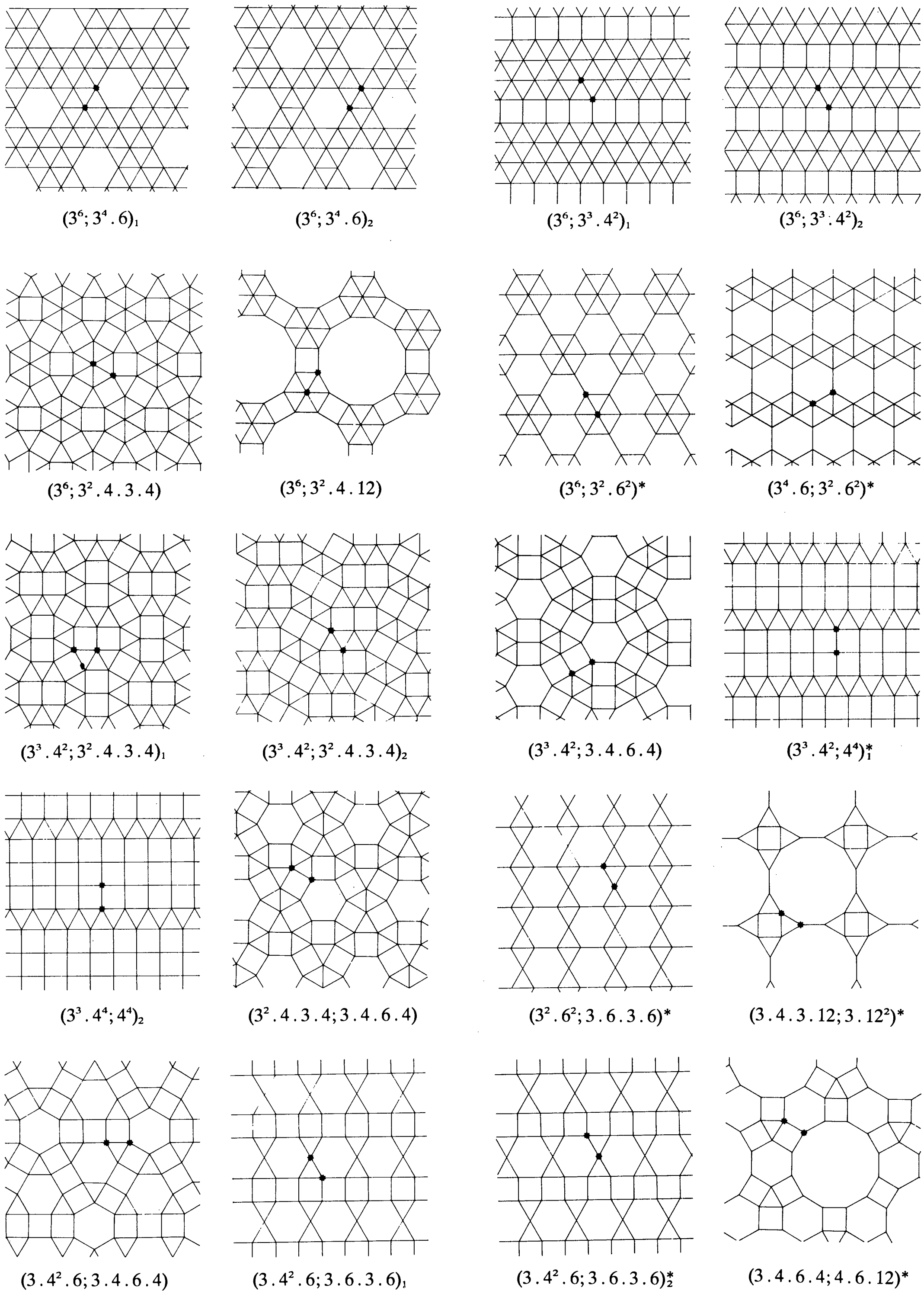
In the same vein, it may easily be verified that the three regular tilings have the following strong transitivity property. If a triplet consisting of a polygon, one of its edges, and a vertex of that edge is called a **flag,** then any two flags of a regular tiling are equivalent under the symmetries of the tiling. From now on, the "regularity" of "regular tilings" will always be understood in this sense, which is becoming more widespread in many related areas; see, for example, Coxeter [1975], Grünbaum [1976]. We should stress that flag-transitivity is more restrictive than requiring that a tiling be vertex-, edge- and tile-transitive: there is exactly one tiling by polygons (FIGURE 16a) which has the latter three kinds of transitivity, but which fails to be regular.

2. *k*-uniform tilings

The observation that the Archimedean tilings are uniform suggests the following possibility of generalization. A tiling is called ***k*-uniform** if its vertices form precisely k transitivity classes with respect to the group of all symmetries of the tiling. Clearly, uniform tilings coincide with 1-uniform tilings. If the types of vertices in the k classes are $a_1.b_1.c_1.\ldots; a_2.b_2.c_2.\ldots;\ldots; a_k.b_k.c_k.\ldots$, we will designate the tiling by the symbol $(a_1.b_1.c_1.\ldots; a_2.b_2.c_2.\ldots;\ldots; a_k.b_k.c_k.\ldots)$, with the obvious shortening through the use of superscripts, and with subscripts to distinguish tilings in which the same types of vertices appear. *There exist* 20 *distinct types of* 2*-uniform edge-to-edge tilings by regular polygons.* They are shown in FIGURE 9. The proof of this fact may be carried out along lines analogous to those explained in connection with the 11 uniform tilings. However, the details are here much more intricate; it appears that the only published version of the proof is found in the paper of Krötenheerdt [1969].

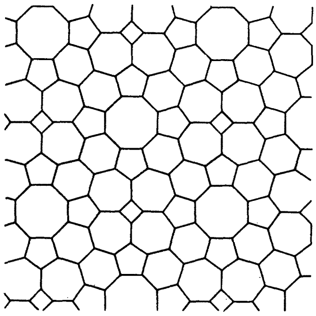
It is not hard to see that there exist k-uniform tilings for each $k \geqq 1$. Examples are given in Krötenheerdt [1969] for $k = 3,4,5,6$ and 7, and in FIGURE 10 for $k = 3,4$. However, even for $k = 3$ it is not known how many distinct 3-uniform tilings exist, nor is any kind of asymptotic estimate available for the number of k-uniform tilings with large k.

A closely related notion was also examined by Krötenheerdt [1969], [1970a], [1970b]. He considered those k-uniform tilings in which the k transitivity classes of vertices consist of k *distinct*



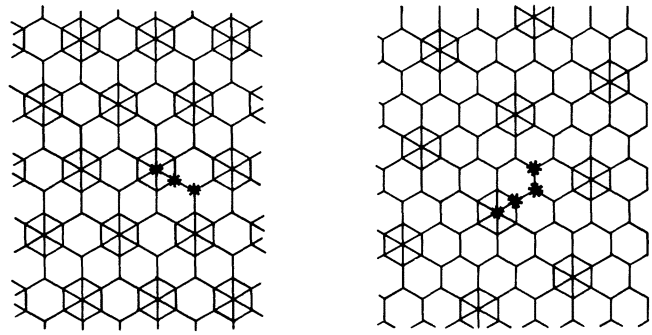
The 20 different types of 2-uniform tilings. The tiling $(3^6; 3^4.6)_2$ exists in mirror-symmetric forms, only one of which is shown. Tilings marked by an asterisk are homogeneous in the sense defined in Section 3. One vertex of each transitivity class is marked.

FIGURE 9



A fake tiling with regular polygons, adapted from a children's coloring book *Altair Design* (Holiday [1970]).

FIGURE 8



Two examples of homogeneous tilings; one is also 3-uniform, the other is 4-uniform. One vertex of each transitivity class is marked.

FIGURE 10

types of vertices. While it is easily seen that for $k = 1$ and for $k = 2$ these coincide with the k-uniform ones, Krötenheerdt's condition is actually restrictive for $k \geqq 3$. Denoting by $K(k)$ the number of distinct Krötenheerdt tilings, he established that $K(1) = 11$, $K(2) = 20$, $K(3) = 39$, $K(4) = 33$, $K(5) = 15$, $K(6) = 10$, $K(7) = 7$ and $K(k) = 0$ for each $k \geqq 8$. Krötenheerdt's method of proof is a natural extension of the one used in the determination of the uniform tilings.

3. Homogeneous and edge-transitive tilings

Departing from the terminology used by some authors, we shall say that an edge-to-edge tiling of the plane by regular polygons is ***k*-homogeneous** if the tiles form precisely k transitivity classes under the symmetries of the tiling. We shall also say that a tiling is **homogeneous** if all tiles that are mutually congruent form one transitivity class. It is easily verified that all the uniform tilings are homogeneous, except $(3^4.6)$, which is 3-homogeneous. Other homogeneous tilings are the seven 2-uniform tilings marked by an asterisk in FIGURE 9.

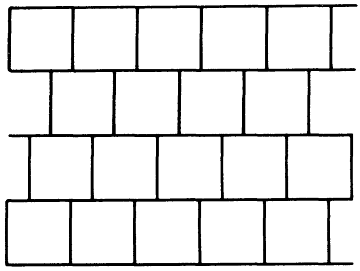
It is rather surprising that there seems to be no consideration in the literature of the homogeneous or k-homogeneous tilings. It appears reasonable to expect that for each k there exists a least number $h(k)$ such that every k-uniform tiling is also h-homogeneous for some $h \leqq h(k)$. Likewise, there probably exists a least number $k(h)$ such that every h-homogeneous tiling is also k-uniform for some $k \leqq k(h)$. From the above remarks and from FIGURE 9 it is easy to see that $h(1) = 3$ and $h(2) = 5$. On the other hand clearly $k(1) = 1$, while the examples of FIGURE 10 show that $k(2) \geqq 4$.

The determination of all 2-homogeneous tilings (all of which are, obviously, homogeneous) should not be very hard, and even the determination of all homogeneous tilings is probably possible with a little patience. We conjecture that the 19 homogeneous tilings shown in FIGURES 7, 9 and 10 are the only ones possible, and that, in consequence, there are just fourteen 2-homogeneous tilings, and that $k(2) = 4$.

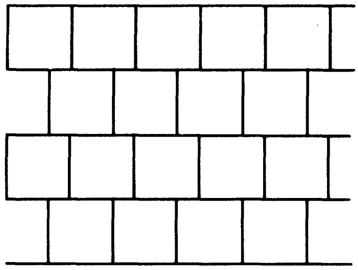
Similar problems arise if we consider transitivity classes of edges. If there are j such classes in a tiling we shall call it a ***j*-edge-transitive tiling.** We mention this idea because we believe that it also is not considered in the literature. There appear to be just four 1-edge-transitive tilings by regular polygons (namely (3^6), (4^4), (6^3) and $(3.6.3.6)$) and four 2-edge-transitive tilings (namely $(3^2.4.3.4)$, $(3.4.6.4)$, (3.12^2) and (4.8^2)).

4. Tilings that are not edge-to-edge

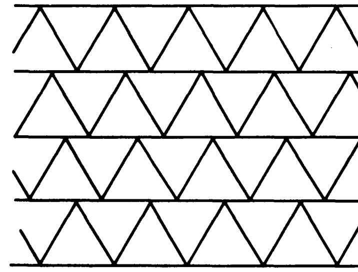
We now consider tilings by regular polygons without the requirement that the tiles meet edge-to-edge. Kepler briefly considered this possibility (see drawings *Bb* and *Kk* in FIGURE 1), but no further consideration seems to have been given to the mathematical possibilities for several centuries.



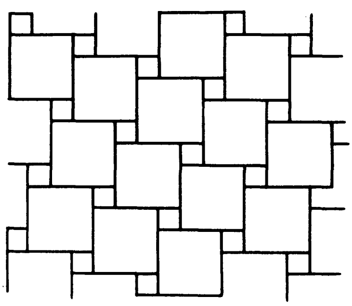
$0<\alpha\leqq\frac{1}{2}$



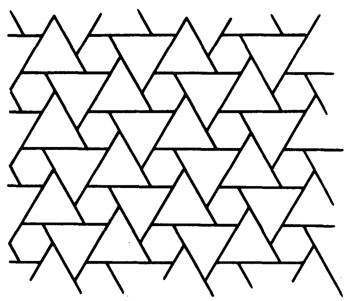
$0<\alpha\leqq\frac{1}{2}$



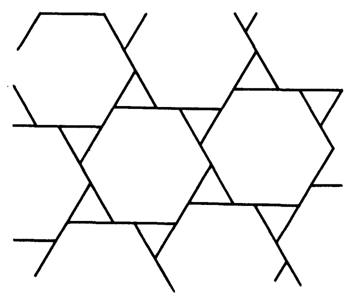
$0<\alpha\leqq\frac{1}{2}$



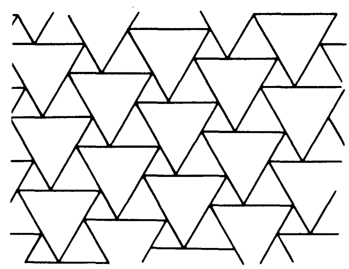
$0<\alpha<1$



$0<\alpha<1$



$0<\alpha<1$



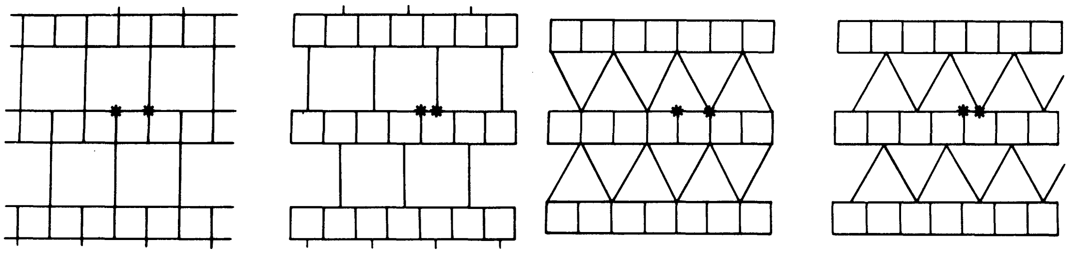
$0<\alpha\leqq\frac{1}{2}$

Representative examples of the seven families of uniform tilings that are not edge-to-edge. Explanation of the parameters is given in Section 4.

FIGURE 11

In a tiling that possibly is not edge-to-edge we shall call each point that is a vertex of some tile a **node** of the tiling, and we shall call a tiling **uniform** provided the symmetries of the tiling act transitively on its nodes. It is not hard to prove that all uniform tilings by regular polygons that are not edge-to-edge may be arranged in seven families, each family depending on a real-valued parameter α. These seven families are illustrated in FIGURE 11. In the first three families each tiling uses only mutually congruent tiles, and the parameter indicates the fraction of overlap between edges of adjacent tiles. The tilings of the next three families use two non-congruent kinds of tiles, and the parameter indicates the ratio of their edge-lengths. In the last family three sizes of triangles appear and the parameter α denotes the ratio of the side of the smallest triangle to that of the largest. (If $\alpha = 1/2$ only two sizes of triangles occur).

It is obviously possible to apply the definitions of k-uniformity, homogeneity, etc., to tilings that are not edge-to-edge. In FIGURE 12 we show several 2-uniform and homogeneous tilings of this kind and it is easy to construct additional examples of a similar character. A complete enumeration of homogeneous, 2-uniform tilings by regular polygons is probably obtainable with moderate effort. Many ornamental designs contain uniform tilings that are not edge-to-edge; see, for example, Dye [1937, FIGURES C15b, K5a, Y2b, Y3a, &a la].

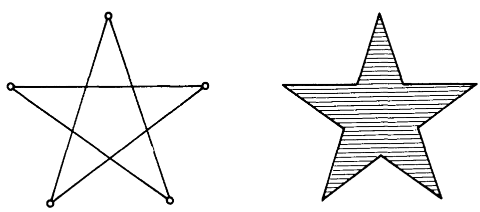


Examples of homogeneous and 2-uniform tilings that are not edge-to-edge. One vertex of each transitivity class is marked.

FIGURE 12

5. Tilings that use star-polygons

The drawings reproduced in FIGURE 1 show that Kepler had a rather pragmatic and experimental approach to tilings. He was looking for various more or less regular tilings, and although his main concern was with tilings that have vertices of a single species, several other possibilities are evident. One such variant, considered by Kepler but apparently not discussed in this form since, deals with edge-to-edge tilings that include star-polygons. In the first book of Kepler [1619] star-polygons are obtained by extending the sides of regular convex polygons. In a rather modern spirit, Kepler treats as vertices of star-polygons only the endpoints of these extended edges, not the vertices of the original convex polygon. Thus the pentagram (FIGURE 13a) has only 5 vertices and five edges. However, when dealing with tilings in Book 2 (and to some extent also later, in connection with the regular non-convex "Kepler–Poinsot" polyhedra), Kepler treats the star-pentagon (FIGURE 13b) as a non-convex decagon which may be called a pentacle and uses other star-polygons in the same way. It



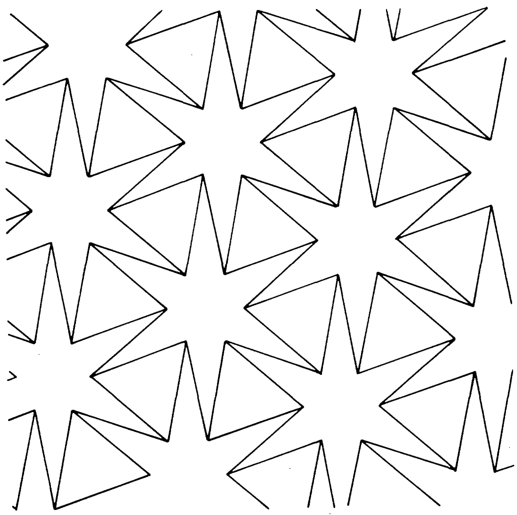
(a) (b)

The two interpretations of regular star-pentagons: (a) the pentagram, consisting of just 5 vertices and 5 edges; (b) the pentacle, a five-pointed "patch" of the plane.

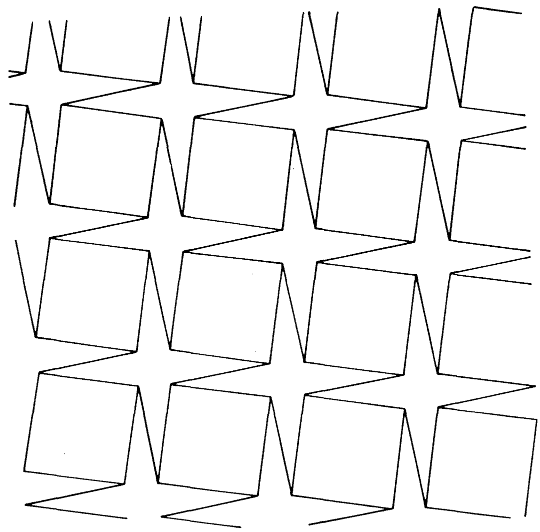
FIGURE 13

is never made quite clear exactly what rules must be followed or what polygons may be used. In Book 1 Kepler speaks only of what today would be called regular star polygons $\{n/d\}$, with n and d coprime. In other words, the edges of the $\{n/d\}$ form a single circuit. In the tiling K of FIGURE 1, however, Kepler not only allows six-pointed stars — which in the regular case would be just "hexagrams", each composed of two triangles — but even permits stars that have angles of $\pi/6$ at their points. At any rate, Kepler missed several possibilities and it is amusing to try to complete his list of tilings under some definite sets of rules.

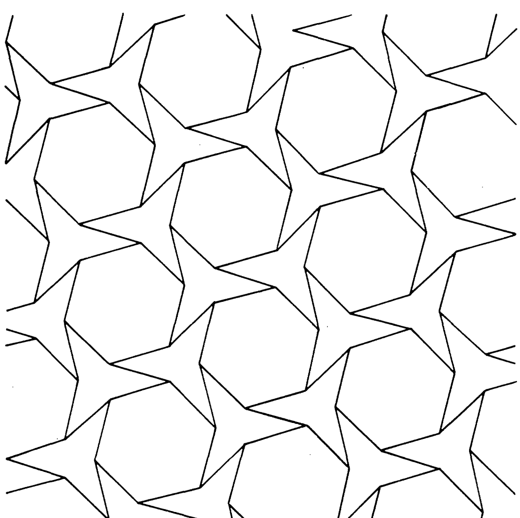
One possibility is to allow as "regular" all n-pointed star-polygons that have the same symmetries as the regular convex n-gon. Such an n-pointed star, denoted $\{n_\alpha\}$ in the sequel, has n vertices of angle α (where $0 < \alpha < (n-2)\pi/n$), n vertices of angle $2(n-1)\pi/n - \alpha$, and $2n$ mutually equivalent edges. Extending the definitions of k-uniform tilings to the case in which regular star-polygons are allowed, it is easy to see that there are precisely four families of 1-uniform tilings, each depending on a real parameter α; these families are illustrated in FIGURE 14. By using single and double asterisks to distinguish between the two kinds of angles in the star-polygons, we can denote these four families by $(3.6^*_\alpha.6^{**}_\alpha)$, $(4.4^*_\alpha.4^{**}_\alpha)$, $(6.3^*_\alpha.3^{**}_\alpha)$ and $(3.3^*_\alpha.3.3^{**}_\alpha)$. (Each of the first three of these families comes in two enantiomorphic forms.) There are many possibilities for 2-uniform tilings that include star-polygons, such as Kepler's *K*, *T*, *Nn*, and those shown in FIGURE 15. Most of these are also homogeneous, if the definition of this term is extended to cover star-polygons. With some patience it should be possible to determine all 2-uniform (and also all homogeneous) tilings that include star-polygons.



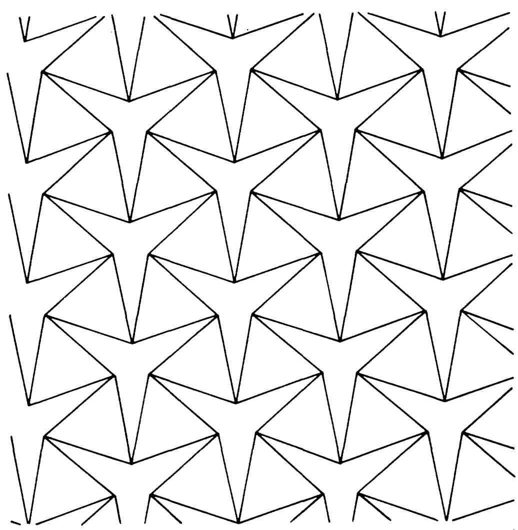
$(3.6^*_\alpha.6^{**}_\alpha)$



$(4.4^*_\alpha.4^{**}_\alpha)$



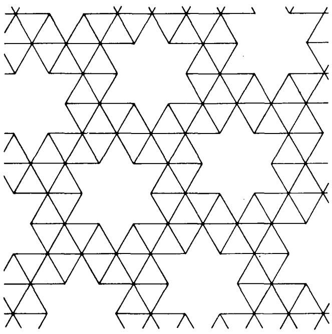
$(6.3^*_\alpha.3^{**}_\alpha)$



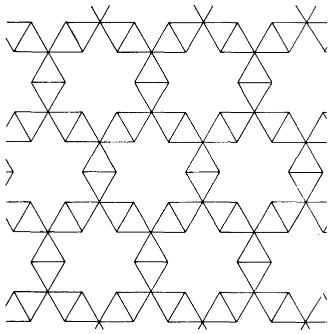
$(3.3^*_\alpha.3.3^{**}_\alpha)$

Representative examples for the four 1-uniform families of tilings that include star-polygons. Explanation of the notation is given in section 5.

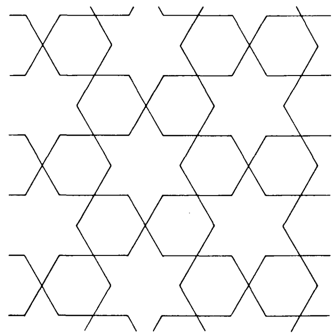
FIGURE 14



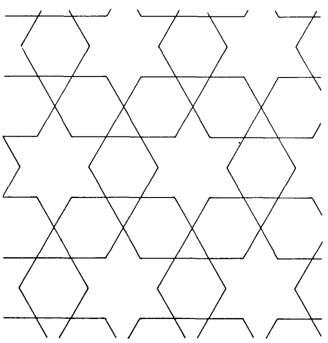
$(3^5 . 6^*_{\pi/3}; 3^2 . 6^{**}_{\pi/3})$



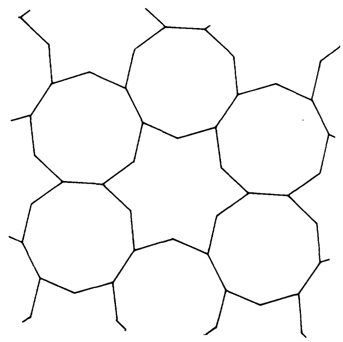
$(3^2 . 6^{**}_{\pi/3}; (3 . 6^*_{\pi/3})^3)$



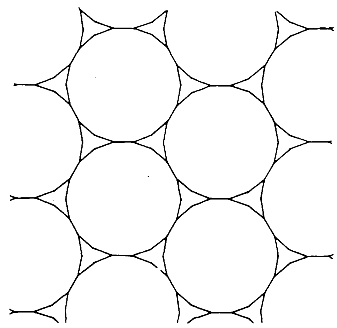
$((6 . 6^*_{\pi/3})^2; 6 . 6^{**}_{\pi/3})$



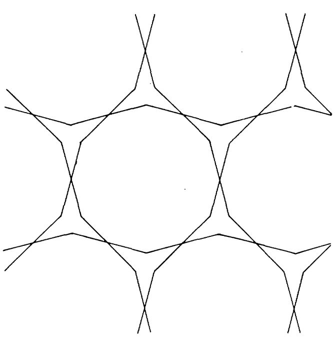
$(3 . 6 . 6^*_{\pi/3} . 6; 6 . 6^{**}_{\pi/3})$



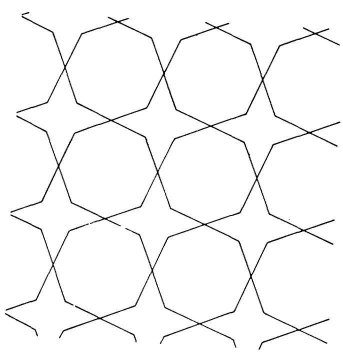
$(9^2 . 6^*_{4\pi/9}; 9 . 6^{**}_{4\pi/9})$



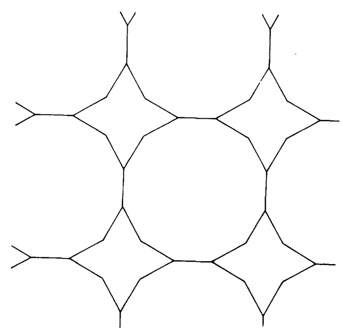
$(18^2 . 3^*_{2\pi/9}; 18 . 3^{**}_{2\pi/9})$



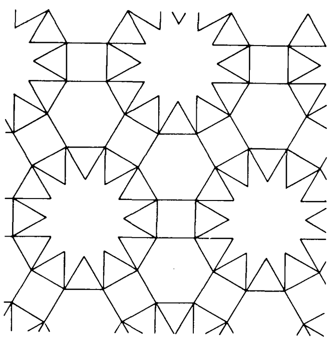
$((12 . 3^*_{\pi/6})^2; 12 . 3^{**}_{\pi/6})$



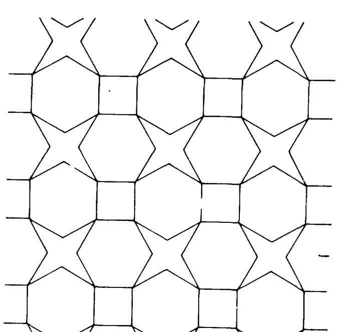
$((8 . 4^*_{\pi/4})^2; 8 . 4^{**}_{\pi/4})$



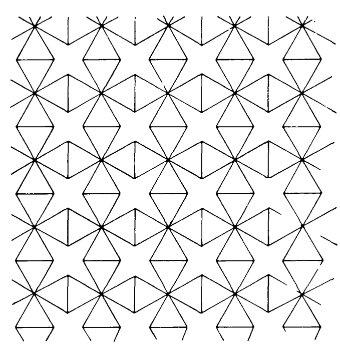
$(12^2 . 4^*_{\pi/3}; 12 . 4^{**}_{\pi/3})$



$(3 . 4 . 6 . 3 . 12^*_{\pi/6}; 3 . 12^{**}_{\pi/6})$



$(4 . 6 . 4^*_{\pi/6} . 6; 6 . 4^{**}_{\pi/6})$



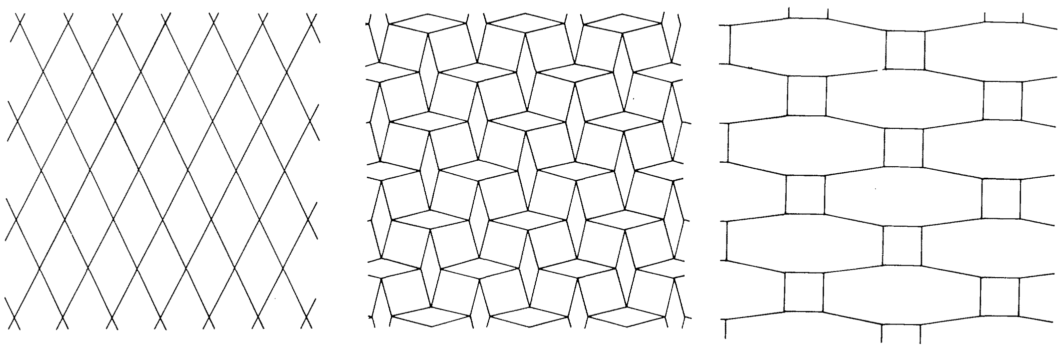
$((3 . 4^*_{\pi/6})^4; 3^2 . 4^{**}_{\pi/6})$

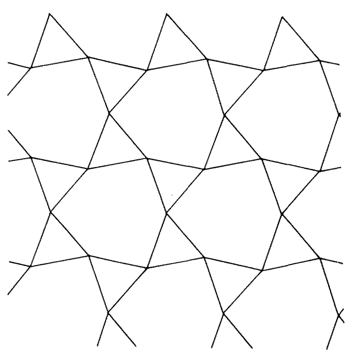
Examples of 2-uniform tilings that include star-polygons. Note the occurrence of 9-gons and 18-gons. All tilings except $(3^5 . 6^*_{\pi/3})$; $3^2 . 6^{**}_{\pi/3})$ are homogeneous.

FIGURE 15

Tilings with star-polygons occur frequently in Islamic art. For example, slightly distorted versions of $(4.4_\alpha^*.4_\alpha^{**})$ may be seen on Plates 104, 117 and 118 in Ipşiroğlu [1971], Plate 45 of which shows the 2-uniform tiling $(3.6.6_{\pi/3}^*.6;\ 6.6_{\pi/3}^{**})$. Similarly, the tiling $(6.6_{\pi/3}^{**};\ 6.6_{\pi/3}^*.6.6_{\pi/3}^*)$ of FIGURE 15 is shown in Plate 1 of Bourgoin [1879]. The tiling $(3^2.4_{\pi/6}^{**};\ 3.4_{\pi/6}^*.3.4_{\pi/6}^*.3.4_{\pi/6}^*.3/4_{\pi/6}^*)$ also occurs as the design of an early American patchwork quilt known as "windmill blades" (Safford & Bishop [1972, FIGURE 173]).

It could be argued that the star-polygons $\{n_\alpha\}$ should actually be called (non-regular) $(2n)$-gons, and that in any case similar treatment should be given to the analogously defined convex (but not regular) polygons $\{n_\alpha\}$ with $(n-2)\pi/n < \alpha < (n-1)\pi/n$ and $n \geqq 2$, in which larger and smaller angles alternate. There is nothing illogical in this suggestion, and it may even not go far enough. It is probably possible (with a reasonable amount of effort) to determine all uniform and homogeneous tilings by arbitrary polygons, in other words, to find all tilings in which several kinds of (not necessarily





Examples of uniform and homogeneous tilings with non-regular polygons. Observe that the first tiling is transitive on vertices, edges, and tiles. All but the last tiling are also transitive on the edges.

FIGURE 16

regular) polygons may be present, but in which all congruent polygons form one equivalence class with respect to the symmetries of the tiling and all vertices are mutually equivalent. Examples of such tilings are shown in FIGURE 16. Even more general problems of a related nature have been considered in the literature, mostly with a crystallographic motivation. For example, attempts were made to determine all tile-transitive tilings (see, for example, Haag [1911], Hilbert & Cohn-Vossen [1932, p. 72], Delone [1959], Heesch [1968]) and all vertex-transitive tilings (Šubnikov [1916], Sauer [1937], Subnikov & Koptsik [1972]), but the claims of success are not justified. Detailed treatments of these questions are given in Grünbaum & Shephard [1977a], [1977b].

6. History

The three regular tilings and several uniform tilings were used as decorations in antiquity and during the Middle Ages; the first mathematical treatment appears to be that of Kepler [1619]. He found all 11 uniform tilings as well as many other kinds of tilings and considered them — in a very modern way — as analogues of the Platonic (regular) and Archimedean polyhedra. (The drawing M in FIGURE 1 does not represent $(3^3.4^2)$ but Kepler's text describes it.) It is therefore strange, almost unbelievable, to find that this part of Kepler's work was completely forgotten for almost 300 years! Although Kepler was frequently quoted by authors interested in regular polyhedra, the first reference to the fact that Kepler determined the 11 uniform tilings appears to be in a note appended by Sommerville to his paper of 1905. Meanwhile, other authors dealt with the topic, usually in connection with investigations of Archimedean or related kinds of polyhedra, but the going was unaccountably slow. Gergonne [1818] obtained several of the uniform tilings; his work was extended, and completeness claimed for the result obtained by Badoureau [1878], [1881]. But although the latter paper is very interesting from several points of view (see Section 7(ii) below) his list of uniform tilings does not contain $(3^4.6)$. Badoureau's defective treatment was uncritically accepted by Lévy [1891] and by Brückner [1900]. The first correct determinations of the 11 uniform tilings in modern times were carried out — independently of each other and blissfully unaware of any of the previous work — by Sommerville [1905] and Andreini [1907]. The proof given by Sommerville that no other uniform tilings are possible is essentially the one we hinted at in Section 1. (The arguments are mentioned also in Ahrens [1901, pp. 66–71], but without a final list of uniform tilings.) Andreini [1907] uses the same method, but in a very cavalier way. He "finds" that there are just 10 (!) possible species of vertices, and the impression is inevitable that he let his "knowledge" of the 11 uniform tilings influence his judgment concerning the possibility of existence of various species. Similarly inadequate is the treatment in Šubnikov [1916]. The proofs or hints given in Kraitchik [1942, p. 203], Bilinski [1948], Critchlow [1970, p. 60], and Williams [1972, p. 42] are similar to the hint given in Section 1. A very nice treatment of this topic and many related questions is given in the refreshingly different text O'Daffer & Clemens [1976]. Several other works present the 11 uniform tilings without proofs (Fejes Tóth [1953, Section 7], [1965, pp. 45–49], Steinhaus [1950, Chapter 4], Cundy & Rollett [1951, Section 2.9]).

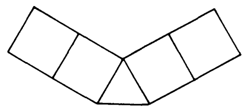
While many erred — as we have seen — in missing some of the uniform tilings, a modern text on architectural design (Borrego [1968, pp. 132, 134]) has too many. It is claimed that the tiling $(3^2.4.3.4)$ exists — like $(3^4.6)$ — in two enantiomorphic forms, not equivalent by motions without reflections!

Many authors have considered ways of generalizing Archimedean tilings by relaxing the requirement that all vertices be of the same type. Actually, even Kepler was interested in such tilings. For example, regarding vertices of species 3 Kepler remarks that they lead to two uniform tilings as well as to the tiling he denotes by O (see FIGURE 1) that "may be continued non-uniformly". His tilings *R*, *Dd*, *Ee* are in our notation the 2-uniform $(3^2.6^2; 3.6.3.6)$, $(3^6; 3^2.4.12)$ and $(3.4.3.12; 3.12^2)$, while his *Cc* may be extended to a 4-uniform $(3^2.4.3.4; 3.4.3.12; 3.4.6.4; 3.4.6.4)$. It is curious that Kepler states that his figure *Kk* cannot be extended without "mixing in" vertices of different species, while actually it appears to be part of the 2-uniform $(3.4^2.6; 3.4.6.4)$, all vertices of which are of species 6.

Kepler did not make precise what kinds of tilings he was interested in, other than the uniform ones. Several later authors were similarly vague, indicating only the desire to limit the species (or the types) of permitted vertices, or trying to obtain more or less symmetric tilings. Such discussions may be found in Lévy [1891], [1894] and especially in Sommerville [1905], while Kraitchik [1942, pp. 205–207] and Steinhaus [1950, Chapter 4] present several examples. Critchlow [1970, p. 60] presents 14 nonuniform tilings and asserts that these are the only possible ones. This assertion is repeated by Williams [1972, p. 43].

As we mentioned in Section 1, not much can be said in way of enumerating all tilings with vertices of just one species. Hence also the papers of Lévy and Sommerville reach no reasonable conclusions. However, there are several lines of investigation that appear to be challenging and promising. They deal with the extension to a tiling of the plane of a given "patch", that is, a finite part of the plane covered by regular polygons without overlaps and without enclosed gaps.

Given a patch such that all the vertices in it are of a species that allows a uniform tiling of the plane, is it always possible to extend the patch to a tiling using only vertices of the same species? Lévy [1891] mentions this question for vertices of species 6; Sommerville [1905] discusses in some detail the possibilities for species 3 and some others.



A "patch" (involving only vertices of species 3) that may be extended at each vertex separately but may not be extended at all vertices simultaneously.

FIGURE 17

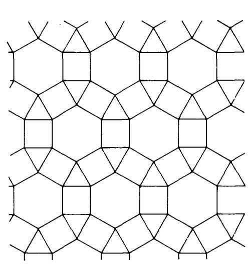
The answer is negative at least in some cases. To see this, we consider the very simple patch in FIGURE 17. Each of its vertices is of species 3, or may be completed to be of species 3. However, such completions may not be carried out simultaneously, so the patch is not part of a tiling with vertices of species 3. Similar examples exist for species 4. The answer is not known for a patch which involves only vertices of species 6. If the answer to this question is affirmative, is it always possible to choose the extension so that the resulting tiling is k-uniform for some k, or to have at least the symmetries of the original patch? Finally, for any variant of these questions, is there an algorithmic decision procedure that would allow the separation of the patches that have extension from the others?

7. Generalizations

We have discussed a number of variants of the theme "more-or-less regular tilings by regular polygons"; nevertheless we have barely scratched the surface of the topic. This final section is devoted to very brief hints at other variants, each of which would deserve a full article (or book) to describe its ramifications.

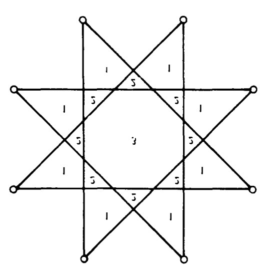
(1) *Multiple tilings.* Petersen [1888] considered the possibility of multiple coverings of the plane by congruent regular polygons of the same kind, subject to the condition that each edge of one tile is an edge of precisely one other. He found that the only way in which this could happen is by superimposing several copies of the regular tilings (3^6), or (4^4), or (6^3). However, it seems that the analogous problem for uniform or Archimedean multiple tilings is still undecided. Moreover, even consideration of multiple tilings with just triangles, or squares, or hexagons — which by Petersen's result consist of superimposed copies of (3^6), (4^4), or (6^3) — leads to the following open question. What multiplicities m are possible in regular tilings? Here "regular" means "flag-transitive", as explained at the end of Section 1. In case of (4^4) probably only $m = k^2$ and $m = 2k^2$ are possible, with integral k; the problem appears to be related to regular maps on the torus (see Coxeter-Moser [1957, Chapter 8]). Multiple tilings by non-regular convex polygons are discussed by Marley [1974].

(ii) *Tiles with densities.* Consider the tiling shown in FIGURE 18. It may be interpreted as the uniform tiling (3 . 4 . 6 . 4) in which every point of the plane which does not lie on an edge is an interior point of exactly one tile. But the same drawing may be interpreted as another uniform tiling, by triangles, hexagons and 12-gons. We must assign a "density" of $+1$ to each hexagon and 12-gon, and a density of -1 to each triangle. With this interpretation the tiling is edge-to-edge, but two tiles sharing an edge may lie on the same side of it (if their densities have different signs). Now each point of the plane (not on an edge) can be assigned a density equal to the sum of the densities of the tiles of which



The uniform tiling (−3 . 12 . 6 . 12) of density 2.

FIGURE 18

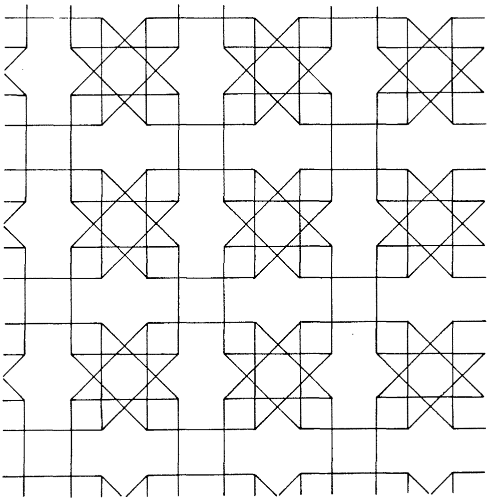


The regular star-polygon {8/3}, with the "density" of the various regions indicated.

FIGURE 19

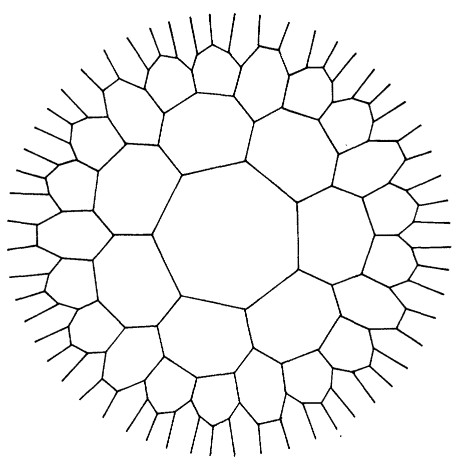
it is an interior point. In this case, it is easy to see that at every such point the density is 2, and so we say that the tiling has density 2. A reasonable symbol for this uniform tiling is $(-3.12.6.12)$.

Analogously, let us consider the regular star-polygon {8/3} (in the modern interpretation, see FIGURE 19), whose 8 vertices are marked by small circles and whose 8 edges connect them in pairs. It is reasonable to assign to the central region a density 3, to the regions near the vertices a density 1, and to the intermediate triangular regions a density 2. Then FIGURE 20 may be interpreted as a uniform tiling $(-4.8.-8/3)$ of density 1, in which each {4} has "density" -1, each {8} density 1, and each {8/3} has densities -1, -2, -3 in its various regions. An approach to tilings related to this (but distinct from it) was followed by Badoureau [1878], [1881] and is also touched upon in Coxeter & Longuet-Higgins & Miller [1954], but there are several complications and a full treatment of such tilings is still not available.



The uniform tiling $(-4.8.-8/3)$ of density 1, first found by Badoureau in 1881.

FIGURE 20



A topologically regular map of type (7^3). Similar topologically regular maps (p^q) exist for all p, q such that $1/p + 1/q < 1/2$.

FIGURE 21

(iii) *Topological uniformity.* In the determination of Archimedean or of uniform convex polyhedra, Euler's equation may be used to provide necessary conditions for the existence of various types. Since many authors treat uniform tilings of the plane together with the Archimedean or uniform polyhedra, the temptation to apply some "limiting form" or "modification" of the Euler relation to such tilings is great. If it were valid such an approach would also have the advantage that it would apply to "topologically uniform" tilings — that is, "maps" in which "symmetries" are not restricted to isometric transformations but may be affected by arbitrary homeomorphisms. However, the execution of such an approach — although feasible — is rather tricky and has to be done with great care (see, for example, Laves [1931], Delone [1959]). The "limiting form" of Euler's theorem usually quoted is, not surprisingly, $V - E + T = 0$, where $V : E : T$ are proportional to the "numbers" of vertices, edges and tiles in the tiling. But this equation only applies to tilings of very restricted kinds. For example, for the "topologically regular map" (7^3) shown in FIGURE 21, in which three heptagonal countries meet at each vertex, it is easy to verify that $V : E : T = 7/3 : 7/2 : 1$, which do not satisfy Euler's theorem. Although the heptagons in this map are not congruent, any two are equivalent under a homeomorphism of the plane that carries the map onto itself. In fact, it deserves the designation "topologically regular" since its self-homeomorphisms act transitively on its flags. While it is easy to verify that there exist "topologically regular maps" (p^q) in which q p-gons meet at each vertex whenever $1/p + 1/q \leqq 1/2$, the question of what "uniform maps" exist is open, as is the question of "Archimedean maps". These two notions are probably distinct; at any rate, while it is easy to verify that no topologically uniform map (3.5^3) exists, the existence of an Archimedean map with all vertices of type 3.5^3 is undecided. These possibilities have escaped many writers, such as Andreini [1907], Šubnikov [1916], Walsh [1972], Loeb [1976, p. 92], who used Euler's relation without due care and reached the conclusion that the only possible "topologically Archimedean maps" of the plane are of the same types as the Archimedean tilings by regular polygons. As the examples of the topologically regular maps show, this is false.

(iv) *Non-Euclidean tilings*. Another variant deserving attention deals with regular convex polygons tiling the sphere or the hyperbolic (non-Euclidean Lobačevski) plane. On the sphere the situation is well known — the uniform or Archimedean tilings may be obtained as central projections of Platonic, uniform, or Archimedean polyhedra. But in this case the distinction between "Archimedean" and "uniform" tilings (or polyhedra) is more than semantic. Besides the uniform (3.4^3) there exists an Archimedean but non-uniform (3.4^3). This appears to have been observed first by Sommerville [1905], and has been rediscovered (often with vehement priority claims) many times since then (Ball [1938, Chapter V], Aškinuze [1957], [1963, p. 430], Lyusternik [1956]). Multiple regular tilings correspond to the Kepler–Poinsot regular non-convex polyhedra, while multiple uniform tilings correspond to polyhedra studied by many authors. See, in particular, Coxeter & Longuet-Higgins & Miller [1954] and Skilling [1975], where references to the earlier literature may be found. Some non-edge-to-edge tilings of the sphere by regular polygons have been considered by Brun [1972]. Digons $\{2\}$ (which are legitimate regular polygons on the sphere) may be used to construct uniform (non-edge-to-edge) tilings of the sphere consisting, for any $n \geqq 3$, of two n-gons and n congruent digons; besides depending on a real-valued parameter, these tilings come in enantiomorphic pairs.

In the hyperbolic plane there exist regular tilings (p^q) whenever $1/p + 1/q < 1/2$ (see, for example, Fejes Tóth [1965, p. 85], Coxeter & Moser [1957, Chapter 5]). There also exist many uniform and Archimedean tilings, but no complete classification is known. This is related to the problem discussed at the end of (iii) above, since the hyperbolic plane is homeomorphic to the Euclidean plane. Detailed consideration of these questions, and partial results, may be found in Bilinski [1948].

(v) *Unbounded polyhedral surfaces.* Finally, as a natural extension of regular or uniform tilings of the plane by regular polygons we may consider the formation of unbounded polyhedral surfaces in 3-dimensional space that are composed of regular convex polygons. Various requirements on the polygons and on the vertices regarding transitivity under symmetries of the surface may be imposed. While it is known (Coxeter [1937]) that only three such surfaces deserve the adjective "regular", there are many that are "uniform", "homogeneous", "Archimedean", etc. Using (p^q) to denote a uniform polyhedral surface in which q regular p-gons $\{p\}$ meet at each vertex, the three regular (so-called Petrie–Coxeter) polyhedra are of types (4^6), (6^4), and (6^6). Uniform polyhedral surfaces are known for types (3^6), (3^7), (3^8), (3^9), (3^{10}), (3^{12}), (4^4), (4^5), (4^6), (5^5), (6^4), and (6^6) (see Gott [1967], Wells [1969], Wachman, Burt & Kleinmann [1974]). Although it is probable that no other types (p^q) are possible, no proof of this conjecture is known. For a discussion of some other related questions see Grünbaum [1977].

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Search Theory: A Mathematical Theory for Finding Lost Objects

A model for determining an optimal search pattern for a wind-driven ship lost at an uncertain position.

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A ship is in distress and you are a Coast Guard search and rescue coordinator whose job is to help save the ship. A last faint radio message from the ship says that due to electrical problems, its navigation equipment has failed and has allowed the ship to wander off course and strike a submerged rock. The ship is taking on water faster than it can be pumped out, and the ship's captain estimates that the boat can stay afloat for only one or two days. He requests extra pumps and a tow to port in order to save the ship. The captain cannot say exactly where he is, but he knows the general region in which he is located.

You have one aircraft available to look for the ship and drop the needed pumps. Considering the number of hours of daylight, the endurance of the aircraft, and the time required to travel to and from the vicinity of the lost ship, you find that there are 10 hours of usable search effort available per day for the next two days. How should these 20 hours of search be spent in order to maximize the probability of finding the ship before it sinks? Which areas should be searched first and for how long? What is the total amount of time that should be spent in each area? These questions can be answered by the use of search theory; since you are a search and rescue coordinator, you can use your remote terminal to interrogate the Coast Guard's central computer to find the answers. Let's see how search theory, via the Coast Guard computer, provides the answers.

First, we must define the problem more carefully by specifying a probability distribution for the target's location and a detection function which relates hours of searching to probability of detection.

Target location distribution

Suppose, for simplicity, that the general region in which the ship is located is rectangular, and subdivided into six rectangular subregions or cells each with area A, as shown in FIGURE 1. Based on your knowledge of the ship's intended route and the captain's description of his location, you feel that cell 1 is distinctly the most likely to contain the ship and the other cells are uniformly less likely to contain it. On this basis, you assign a subjective probability $p(1)=.5$ to the target being in cell 1 and $p(j)=.1$ to the target being in cell j for $j\neq 1$. Within each cell, you assume the ship's location distribution is uniform.

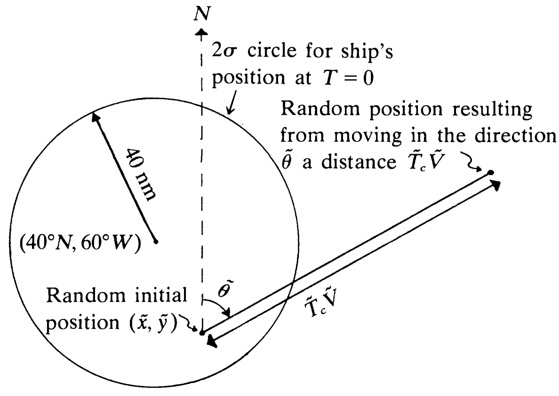
Devising a probability distribution for a target's location is an art rather than a science. The procedure relies on the subjective judgment and experience of the search planner aided by the facts available, such as the last known position, the accuracy of the navigation system employed, and the intended route of the missing object. Indeed, construction of the target location distribution is often the most crucial step in preparing a search plan. For this reason, the Coast Guard central computer contains a number of programs which assist the search planner in constructing these distributions. These programs are designed to account for the uncertainty in the location information and to update past information to the present time by allowing for drift caused by wind and ocean currents.

As an example of a more realistic prior target location distribution, let us suppose that a ship was last reported at 40°N, 60°W heading east at 10 knots. The ship was expected to report its position once a day, but for two days nothing was heard from it. Let $T = 0$ correspond to the time of the last reported position. We shall assume a casualty has occurred some time between $T = 0$ and $T = 24$ hours, since otherwise we would have expected to hear from the ship at $T = 24$ hours. A probability distribution for the target's location could be constructed as follows. If the navigational equipment used by the ship has a circular normal error with standard deviation 20 nm in any direction, then we can use that distribution for the target's location at $T = 0$. That is, let us take the reported position to be the origin of a two-dimensional plane which locally represents the surface of the earth. Let the y-axis be oriented north-south and the x-axis east-west. Then the x-coordinate of the actual position at $T = 0$ is normally distributed with mean 0 and standard deviation 20 nm. Similarly, the y-coordinate has an independent normal distribution with mean 0 and standard deviation 20 nm.

Cell 1 $p(1) = .5$	Cell 2 $p(2) = .1$	Cell 3 $p(3) = .1$
Cell 4 $p(4) = .1$	Cell 5 $p(5) = .1$	Cell 6 $p(6) = .1$

Search region and cells

FIGURE 1



A sample replication

FIGURE 2

Although the reported heading is due east at 10 knots, we assume that the ship's actual heading is uniformly distributed between 80° and 100° and that its speed is uniformly distributed between 8 and 12 knots. Finally, since we do not know the time T_c at which the casualty occurred, we assume that T_c is uniformly distributed between 0 and 24 hours.

A probability map for the ship's position at the time of the casualty can be generated by a Monte Carlo technique as follows. For each replication, random draws are made to determine the ship's initial position $(\tilde{x}, \tilde{y})$, the ship's heading $\tilde{\theta}$, speed $\tilde{V}$, and the length of time $\tilde{T}_c$ to casualty. The ship's position is then advanced from its initial position $(\tilde{x}, \tilde{y})$ in the direction $\tilde{\theta}$ a distance $\tilde{T}_c\tilde{V}$ as shown in FIGURE 2. The resulting position represents one replication. This process is repeated 10,000 times, to obtain the probability distribution shown in FIGURE 3. The unknown length of time till the distress occurred results in a distribution which is elongated in the east-west direction compared to the circular normal distribution for the target's last known position. In the Coast Guard programs, these points are further moved to account for the action of winds and currents on a vessel adrift.

Although we could pursue the above example to show how optimal allocations of effort are obtained, for the purpose of presenting the essentials of search therory we shall return to the simple distribution in FIGURE 1. To make use of this (or any) distribution for the target's location, we must determine a function that relates effort spent to probability of detection. Such a function is called, naturally enough, a **detection function**; we turn now to a study of its properties.

	W 62D OM – W 61D OM	W 61D OM – W 60D OM	W 60D OM – W 59D OM	W 59D OM – W 58D OM	W 58D OM – W 57D OM	W 57D OM – W 56D OM	W 56D OM – W 55D OM	W 55D OM – W 54D OM	W 54D OM – W 53D OM	W 53D OM – W 52D OM
N42D OM – N41D OM			2	5	6	15	29	15	5	
N41D OM – N40D OM	5	176	804	944	972	905	749	309	39	1
N40D OM – N39D OM	1	160	789	994	958	978	737	282	47	
N39D OM – N38D OM			2	7.	6	12	20	23	3	

Probability map of distress locations (Probabilities are multiplied by 10,000.)

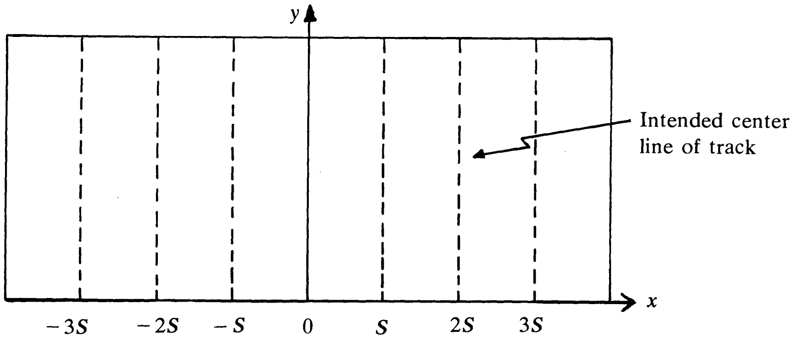
FIGURE 3

The detection function

To obtain a detection function for this example, we assume that there is a detection range d such that the probability of detection is β if the aircraft passes within distance d of the ship, and 0 otherwise. Let $W = 2\beta d$ be the sweep width of the search, and $M = 2d$ be the width of the search path. The sweep width W is a measure of the sensor's average detection capability in the sense that if the search vehicle were to travel in a straight line for a distance l through a region of area A in which there was a uniform distribution of targets all having the same detection characteristics as the ship, then Wl/A fraction of these targets would be detected.

When the aircraft searches one of the cells or rectangles, the pilot will attempt to follow a parallel path search with intended spacing S between tracks as shown in FIGURE 4. Suppose that $\beta = 1$, i.e., the probability of detection is 1 if the aircraft passes within distance d of the ship, that $S = M$, and that the aircraft can follow the intended tracks exactly. In this case the probability of detection as a function of search time t, assuming the target is in the rectangle, would follow the straight line in FIGURE 5.

However, two factors intervene to make the actual detection probability fall below this line. First, experience shows that because of fatigue, boredom, and distractions, an observer can pass within visual sighting range of a target and still not see it. Thus, normally, we are dealing with a situation in which $\beta < 1$. Second, there are usually substantial navigational errors involved in the placement of the tracks, especially when flying over the ocean. It is not uncommon for the search aircraft to have navigational uncertainties with a standard deviation as large as 10 miles (see section 612 of **[8]**). The result of these navigation or track placement errors is to cause overlaps in the coverage of some paths and gaps in others. Thus, even when $\beta = 1$, the probability of detection falls below the straight line in FIGURE 5.



Nominal track placement for parallel path search

FIGURE 4

In order to obtain detection probabilities for a parallel path search, we use the following simple model developed by R. K. Reber [**4**]. Suppose that the search tracks are indeed parallel to the y-axis as shown in FIGURE 4 and that iS is the intended value of the x-coordinate of the ith path. However, the actual value $\tilde{x}_i$ is normally distributed with mean iS and standard deviation σ.

Let $g_i(x)$ be the probability that the point (x, y) in the rectangle is covered by the ith path. Then

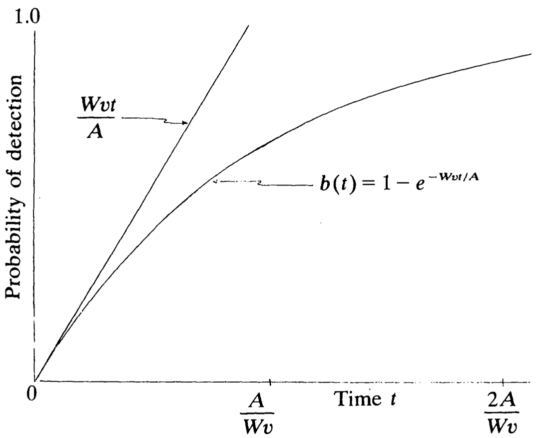
$$g_i(x) = Pr\{\tilde{x}_i - d \leqq x \leqq \tilde{x}_i + d\} = Pr\{x - d \leqq \tilde{x}_i \leqq x + d\}$$

(1)

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{x-d}^{x+d} \exp[-(u - iS)^2/2\sigma^2]du = \frac{1}{\sqrt{2\pi}} \int_{(x-iS-d)/\sigma}^{(x-iS+d)/\sigma} e^{-\frac{1}{2}z^2} dz.$$

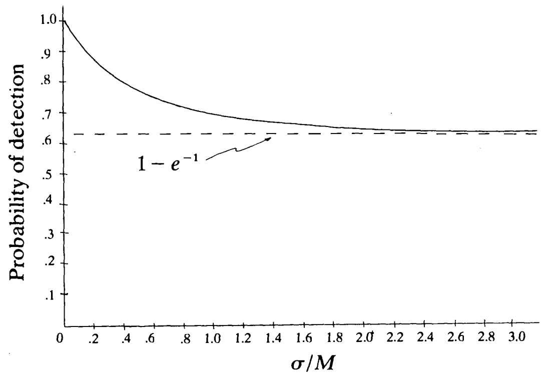
Since the target will be detected with probability β if it falls within the sensor path, $\beta g_i(x)$ is the probability that the target will be detected by the ith track given it is located at (x, y).

Assuming that each path presents an independent chance at detecting the target, the probability of detecting the target on at least one of the $2N + 1$ paths covering the rectangle given it is located at (x, y) in the rectangle is $h(x) = 1 - \Pi_{i=-N}^{N}[1 - \beta g_i(x)]$. For the remainder of this calculation, we shall assume that x is sufficiently far from the boundaries of the rectangle that we can replace the finite



Detection probability for random search

FIGURE 5



Detection probability for parallel path search

FIGURE 6

product in the above equation by the following infinite one so that

$$h(x) = 1 - \prod_{i=-\infty}^{\infty} [1 - \beta g_i(x)]. \tag{2}$$

Ignoring edge effects, one can see from (1) and (2) that the function h is periodic with period S equal to the spacing of the tracks. Thus, the average probability of detection over the rectangle (ignoring edge effects) is

$$Q\left(\frac{M}{\sigma}, \beta, \frac{S}{\sigma}\right) = \frac{1}{S}\int_{-S/2}^{S/2} h(x)dx = \frac{\sigma}{S}\int_{-S/2\sigma}^{S/2\sigma} h(\sigma y)dy. \tag{3}$$

Observe that by the way we have written equation (3), we are claiming that Q is a function of only M/σ, β, and S/σ. By referring back to equations (1) and (2), the reader may check that this is true.

The probability Q is plotted as a function of $(M/\sigma)^{-1}$ in FIGURE 6 for the case where $\beta = 1$ and $M = S$. Notice that as M/σ becomes small, Q approaches $1 - e^{-1} \approx 0.63$. In fact, it can be shown that for arbitrary M, β, and S,

$$\lim_{\substack{M/\sigma \to 0 \\ S/\sigma \to 0}} Q\left(\frac{M}{\sigma}, \beta, \frac{S}{\sigma}\right) = 1 - e^{-\beta M/S}, \tag{4}$$

provided M/S remains constant. From FIGURE 6 we can see that this limiting case is approached quite quickly when $\beta = 1$ and $M = S$. Numerical experience indicates that when $\beta M/\sigma$ and $S/\sigma \leqq \frac{1}{2}$, equation (4) gives a good and somewhat conservative approximation to Q. Since we have already noted that one can have navigational errors with standard deviations of 10 miles or greater and since in our example we will be dealing with visual search with $\beta M = 3.2$ miles, we shall assume from here on that the limit in (4) holds. In practice, one usually chooses $S \leqq M$ so that there are no intentional gaps in the coverage.

For our purposes it is most convenient to restate (4) in terms of the time t spent in the search area. Suppose the search rectangle has length a_1 and width a_2. Then $A = a_1 a_2$ is the area of the rectangle and $A/S = a_1 a_2/S$ is the track length required for parallel path search of the rectangle with tracks spaced a distance S apart. Suppose that the aircraft travels at an average speed of v; then $t = A/Sv$ is the time spent performing such a search. Recalling that $W = \beta M$, we have $\beta M/S = W/S = Wvt/A$ and $b(t)$, the probability of detection by time t, becomes

$$b(t) = 1 - \exp(-Wvt/A) \quad \text{for } t \geqq 0. \tag{5}$$

This is the random search formula of B. O. Koopman which was originally obtained in **[3]** by a heuristic but well motivated argument. Experience has shown that the exponential detection function defined in (5) provides reasonable and conservative estimates of detection capability in many search situations. This fact and its simple form account for its widespread use in search problems. For systematic searches of an area, reality is bounded by the two curves in FIGURE 5.

An important property of the random search formula is shown in FIGURE 5. Initially, the slope of b is Wv/A since at the beginning of the search there is almost no chance of overlapping with previous search. However, as the search continues, the chance of overlap with previously searched areas of the rectangle increases, and this slows the rate at which the probability of detection increases. Consequently, the random search formula exhibits a diminishing rate of return. This diminishing rate (or decreasing derivative) of b will play an important role in finding the optimal allocation in the next section.

Optimal allocation of effort

We can now find the optimal allocation of effort for rescuing the sinking ship. Suppose that the boat is 50 feet long, that each cell is a 50–mile by 50–mile square and that the meteorological visibility is three miles over the entire region. (Meteorological visibility is the maximum distance at which very large objects such as mountains can be seen.) Checking Figure 7–2 of [**8**] we find that if the aircraft flies at the optimum height for this situation, it will have a visual sweep width of $W = 3.2$ nm. The aircraft searches at a speed of 100 knots, so $A = 2500\ (\text{nm})^2$, $v = 100$ knots, and $W = 3.2$ nm.

If the aircraft spends time t_i searching in cell i for $i = 1, \ldots, 6$, then the probability of detecting the ship with this allocation of time is

$$\sum_{j=1}^{6} p(j)b(t_j) = \sum_{j=1}^{6} p(j)(1 - e^{-Wvt_j/A}).$$

We shall assume that the time required to switch from one cell to the other is negligible, so that the total time (or cost) required by this allocation is $\sum_{j=1}^{6} t_j$. In search problems, the time required to complete a search is often taken to be the cost of the search.

More generally, let f be an allocation of time, i.e., $f(j) \geqq 0$ is the amount of time spent looking in cell j for $j = 1, \ldots, 6$. Define

$$P[f] = \sum_{j=1}^{6} p(j)b(f(j)), \tag{6}$$

$$C[f] = \sum_{j=1}^{6} f(j). \tag{7}$$

Then $P[f]$ gives the probability of detection and $C[f]$ gives the cost (or total time) associated with the allocation f. Let us consider the optimal allocation of K hours of search. That is, we attempt to find an allocation f^* such that $C[f^*] \leqq K$ and

$$P[f^*] = \text{maximum of } P[f] \text{ over all allocations } f \text{ such that } C[f] \leqq K. \tag{8}$$

Such an f^* is called optimal for cost K.

Naively, one might suppose that, since the cells are of equal size and the sweep width is equal in all cells, one should allocate his effort in proportion to the probability $p(j)$ of the target being in cell j for $j = 1, \ldots, 6$. However, as we shall see below, this does not produce an optimal allocation. The reason for this lies in the decreasing rate of return exhibited by the detection function b.

Suppose the aircraft has spent time t_j looking in cell j for $j = 1, \ldots, 6$ and is considering spending a small increment of time h looking in cell j. The increase in probability of detection resulting from this increment is approximately $p(j)b'(t_j)h$. In the short run, the searcher would benefit most from placing the increment in the cell having the highest value of $p(j)b'(t_j)$.

Define $\rho_j(t) = p(j)b'(t)$ for $j = 1, \ldots, 6$ and $t \geqq 0$. Then ρ_j is called the rate of return function for cell j. Because the function b exhibits a decreasing rate of return, the rate ρ_j is similarly decreasing. Let us examine the search policy which always places the next small increment of effort in the cell having the highest rate of return. This rate will depend on t_j, the time previously spent looking in cell j for $j = 1, \ldots, 6$. Under this policy, the next increment goes into the cell j^* that satisfies $\rho_{j^*}(t_{j^*}) = \max\{\rho_j(t_j) \mid j = 1, 2, \ldots, 6\}$. Such a search policy is called **locally optimal.** We shall show below that following a locally optimal policy for t hours yields an optimal allocation of t hours of effort for all t; we will do this in particular for $t = 10$ and $t = 20$ hours.

At time 0 when no search has been made in any cell, $\rho_j(0) = p(j)b'(0) = p(j)Wv/A$. Since $p(1) = .5$ and $p(j) = .1$ for $j \neq 1$, the locally optimal policy calls for looking solely in cell 1 until the time s that satisfies

$$\rho_1(s) = .5\,\frac{Wv}{A}\,e^{-Wvs/A} = .1\,\frac{Wv}{A} = \rho_j(0) \text{ for } j \neq 1,$$

i.e.,

$$s = \ln 5\,\frac{A}{Wv} = 7.81 \ln 5 \text{ hrs} = 12.6 \text{ hrs}.$$

In order to continue the locally optimal policy beyond search time $s = 12.6$ hrs., the additional time h_j spent looking in cell j for $j = 1, \ldots, 6$, must be split so that $\rho_1(s + h_1) = \rho_j(h_j)$ for $j \neq 1$. In other words,

$$\rho_1(s + h_1) = .5\frac{Wv}{A}e^{-Wv(s+h_1)/A} = .1\frac{Wv}{A}e^{-Wvh_1/A} = .1\frac{Wv}{A}e^{-Wvh_j/A} = \rho_j(h_j),$$

which implies $h_j = h_1$, for $j = 1, \ldots, 6$.

This search policy may be described as follows: Let $\varphi^*(j, t)$ be the number of hours out of the first t that are spent looking in cell j for $j = 1, \ldots, 6$. Then

$$\varphi^*(1, t) = \begin{cases} t & \text{for } 0 \leqq t \leqq 12.6 \\ 12.6 + \frac{1}{6}(t - 12.6) & \text{for } 12.6 < t < \infty \end{cases}$$

(9)
$$\varphi^*(j, t) = \begin{cases} 0 & \text{for } 0 \leqq t \leqq 12.6 \\ \frac{1}{6}(t - 12.6) & \text{for } 12.6 < t < \infty,\ j = 2, \ldots, 6. \end{cases}$$

Following the locally optimal plan φ^* would require splitting effort instantaneously and uniformly among the six cells if the search progressed past 12.6 hours. Obviously the aircraft cannot do this. However, if the search planner has t hours of search effort available, he can calculate $\varphi^*(j, t)$, the total search time that accumulates in cell j during the time interval t. He could then allocate the aircraft's search effort so that it spends $\varphi^*(j, t)$ hours searching in cell j for $j = 1, \ldots, 6$. We now show that if the planner does this for $t = 10$ and $t = 20$ hours, he will obtain an optimal allocation for 10 and 20 hours of search respectively.

To show that one obtains an optimal allocation for 10 hours, let $f^*(1) = \varphi^*(1, 10) = 10$ and $f^*(j) = \varphi^*(j, 10) = 0$ for $j = 2, \ldots, 6$, so that f^* gives the allocation resulting from following the plan φ^* for 10 hours. While the plan φ^* has the virtue of always adding search effort to maximize the short-term gain, it is not obvious that such a policy produces an optimal allocation of the total effort available. It is conceivable that a plan that considered the total 10 hours of search available for the first day might produce a higher probability of detection at the end of this 10 hours than that produced by following φ^*. However, by using Lagrange multipliers we can show that f^* is optimal for cost 10 hours.

Consider the function l defined by

(10)
$$l(j, \lambda, t) = p(j)\,b(t) - \lambda t \text{ for } j = 1, \ldots, 6,\ \lambda \geqq 0, \text{ and } t \geqq 0.$$

This function is called the pointwise Lagrangian, and λ is called a Lagrange multiplier. Let $\lambda = \rho_1(f^*(1))$. Then by the nature of the locally optimal plan, $\lambda \geqq \rho_j(f^*(j)) = \rho_j(0)$ for $j = 2, \ldots, 6$.

We shall now show that

(11)
$$l(j, \lambda, f^*(j)) = \text{maximum of } l(j, \lambda, t) \text{ over } t \geqq 0 \text{ for } j = 1, \ldots, 6.$$

For $j = 1$, this is accomplished by taking the derivative l' of l with respect to t, recalling that b' is decreasing, and observing that

$$l'(1, \lambda, t) = \rho_1(t) - \lambda = p(1)b'(t) - \lambda \begin{cases} \geqq 0 \text{ for } 0 \leqq t \leqq f^*(1), \\ \leqq 0 \text{ for } f^*(1) < t < \infty. \end{cases}$$

For $j = 2, \ldots, 6$, we find that $l'(j, \lambda, t) = \rho_j(t) - \lambda \leqq 0$ for $t \geqq 0$, so that the maximum of $l(j, \lambda, \cdot)$ occurs at $t = 0$. This proves (11).

The fact that f^* satisfies (11) allows us to show that f^* is optimal for cost $K = 10$ hours as follows. Suppose that f is an allocation such that $C[f] = K = 10$ hours. Then by (11),

$$l(j, \lambda, f^*(j)) \geqq l(j, \lambda, f(j)) \text{ for } j = 1, \ldots, 6. \tag{12}$$

Summing both sides of (12) on j and making use of (6), (7), and (10), one obtains $P[f^*] - \lambda C[f^*] \geqq P[f] - \lambda C[f]$, which implies $P[f^*] - P[f] \geqq \lambda(C[f^*] - C[f]) = 0$, where the last equality follows from $C[f^*] = C[f]$. Thus, f^* is optimal for cost $K = 10$ hours. One can check that $P[f^*] = .36$. (Observe that this optimization cannot be performed by using the standard Lagrange undetermined multiplier method, which involves solving for a critical point, because the optimum lies on the boundary of the domain of the function being maximized.)

We now know how to allocate the first day of search, but what if the search lasts beyond that? Returning to the plan φ^*, we see that by setting $\lambda = \rho_1(\varphi^*(1,20))$, we can follow the same argument as above to show that $\varphi^*(1,20) = 13.83$, and $\varphi^*(j,20) = 1.23$ for $j = 2, \ldots, 6$, is optimal for cost $K = 20$ hours.

This is a fortunate result since it means that the optimal plan for 20 hours of search can be treated as a continuation of the optimal plan for 10 hours. The probability of detecting the target by the time 20 hours of search is completed is .49.

Uniformly optimal plans

A retrospective glance at the proof given above shows that for any $t \geqq 0$, $\varphi^*(\cdot, t)$ is optimal for cost t. We say, in this case, that φ^* is a **uniformly optimal** plan. Such a plan avoids the problem of sacrificing some probability of detection at intermediate times in order to obtain the maximum detection probability by 20 hours of search. However, as we noted above, it is not possible to follow the plan φ^* exactly. Instead, the coordinator will have to settle for some operationally reasonable approximation to the uniformly optimal plan if it is important to maintain high intermediate values of probability of detection. In the case of the sinking ship, it is indeed desirable to try to follow the uniformly optimal plan since the captain's figure of two days until the ship sinks is at best an estimate and could be optimistic.

Let us return to the plan that allocates effort in proportion to the prior probabilities $p(j)$ for $j = 1, 2, \ldots, 6$, and see how it compares with the optimal plan. The table in Figure 7 compares the probability of detection which results from following the proportional plan and the optimal plan.

Hours of Search	Optimal Plan	Proportional Plan	Percent Improvement of Optimal Plan over Proportional Plan
2	.11	.07	57
4	.20	.14	43
6	.27	.20	35
8	.32	.25	28
10	.36	.30	20
12	.39	.34	15
14	.42	.38	11
16	.44	.41	7
18	.47	.44	7
20	.49	.47	4

Probability of detection

Figure 7

Notice that the optimal plan yields the biggest improvement in the early hours of the search. This is important because early success increases the chance of rescuing a lost person alive, or of saving a ship before it goes down and decreases the cost of the search effort.

Usually the Coast Guard must deal with more complicated distributions than the one considered for the example. Often there are a large number of cells (e.g., 50) and it is commonplace for the sweep width to vary from cell to cell because of variations in visibility. Example 2.2.8 of [7] gives an algorithm for computing optimal allocations for an n-region target distribution and an exponential detection function which may vary from region to region. A similar algorithm is employed by the Coast Guard's computer-assisted search planning (CASP) system which is discussed further in [5].

Uniformly optimal search plans have been found for a wide variety of searches involving stationary targets. When the detection function has a continuous, positive and strictly decreasing derivative, the detection function is called regular. For regular detection functions, Theorems 2.2.4 and 2.2.5 of [7] give uniformly optimal search plans in a form readily adapted to computer calculation. For example, suppose there are J regions and that $p(j)$ is the probability that the target is in region j. Assume that the detection function b is regular. For $j = 1, \ldots, J$, let $\rho_j(t) = p(j)b'(t)$ for $t \geqq 0$, and let $\rho_j^{-1}(\lambda)$ be the inverse of ρ_j evaluated at λ when $0 < \lambda \leqq \rho_j(0)$ and 0 when $\rho_j(0) < \lambda$. Define $U(\lambda) = \sum_{j=1}^{J} \rho_j^{-1}(\lambda)$ for $\lambda > 0$ and $\phi^*(j,t) = \rho_j^{-1}(U^{-1}(t))$ for $t \geqq 0$, $j = 1, 2, \ldots, J$. Then ϕ^* is uniformly optimal among all plans ϕ which satisfy $\sum_{j=1}^{J} \phi(j,t) = t$ for $t \geqq 0$. The functions ρ_j and U and their inverses are easily calculated numerically. Occasionally, as in the case of a circular normal target distribution coupled with an exponential detection function, these functions can be computed analytically; see examples 2.2.1 and 2.2.7 of [7]. One might hope that uniformly optimal plans exist for a wide class of moving target problems in the same fashion as they do for stationary target problems. However, uniformly optimal plans are the exception rather than the rule for moving targets.

Search theory is a relatively new area of applied mathematics. It had its beginning during the years 1942–1945 in the work done by Bernard O. Koopman and his colleagues in the Anti-Submarine Warfare Operations Research Group of the U.S. Navy. A compendium of this work [3] was written in 1946 and still remains one of the basic references in the area. In fact, Koopman is in the process of revising and updating this work. In the years since 1946, the subject has grown and matured into a field with an extensive body of results. Yet the field still retains its motivation in terms of actual problems which suggest areas of research. For example, the 1968 search for the U.S. nuclear submarine Scorpion (see [6]) suggested work on search with uncertain sweep width and search in the presence of false targets. Work for the U.S. Coast Guard to develop computer-assisted search-planning programs suggested the study of conditionally deterministic motion, discussed in Chapter 8 of [7]. The interested reader can gain an introduction to a wide range of problems in search and detection theory by perusing [1] and [2], the published bibliographies on search and detection theory.

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Velocity Averages

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A particle travels on a straight line from time $t = a$ to $t = b$. What is its average velocity? If $f(t)$ denotes the position of the particle at time t, the conventional answer is $(f(b) - f(a))/(b - a)$. If f is differentiable, one can also consider the average of the velocities, $(f'(a) + f'(b))/2$, or the velocity at the average time, $f'((a + b)/2)$. Beginning students of calculus often confuse the latter two averages with the first. Their problems are exacerbated by the fact that for a uniformly accelerated particle the results are identical. (This follows easily from the general form $f(t) = At^2 + Bt + C$ for motion under uniform acceleration.) The purpose of this note is to show that these three averages are equal *only* in the case of uniform acceleration.

We will actually prove a bit more: *If any two of the three expressions — average velocity, average of the velocities, and velocity at the average time — agree on each time interval* $[a, b]$, *then* f'' *exists and is constant.*

First, suppose that the average velocity equals the average of the velocities on each interval $[c, d]$. Then

$$(f(t) - f(a))/(t - a) = (f'(a) + f'(t))/2 \quad (t \neq a). \tag{1}$$

Differentiation of (1) shows that $f''(t)$ exists for every $t \neq a$. Since a is arbitrary, $f''(t)$ exists for every t. Similar reasoning shows that f has derivatives of all orders.

From (1) we conclude that

$$2(f(t) - f(a)) = (t - a)(f'(a) + f'(t)) \tag{2}$$

for all real numbers a and t. By differentiating (2) twice, we discover that $(t - a)f'''(t) = 0$. Hence, $f'''(t) = 0$ for every $t \neq a$. Since a is arbitrary, $f'''(t) = 0$ for every t; and f'' is constant.

Here is an alternative proof based on the chord trapezoidal rule of approximate integration. This rule says that, whenever g is twice differentiable on an interval $[a, b]$, there exists a point s such that $a < s < b$ and

$$\int_a^b g(t)dt = (1/2)(b - a)[g(a) + g(b)] - (1/12)(b - a)^3 g''(s). \tag{3}$$

Setting $g = f'$ and using the fundamental theorem of calculus to evaluate the left side of (3), we obtain

$$f(b) - f(a) = (1/2)(b - a)[f'(a) + f'(b)] - (1/12)(b - a)^3 f'''(s).$$

It follows from (1) (with $t = b$) that in this case the chord trapezoidal approximation is exact, that is, the "error" term, $-(1/12)(b - a)^3 f'''(s)$, must vanish. Thus, in each interval $[a, b]$ there exists a point s such that $f'''(s) = 0$. Since f''' is continuous, f''' is identically zero, as desired.

A variation on this argument can be used to verify the second case in our three-part proposition. The tangent trapezoidal rule states that, if g is twice differentiable on the interval $[a, b]$, then there exists a point s such that $a < s < b$ and

$$\int_a^b g(t)dt = (b-a)g((a+b)/2) + (1/24)(b-a)^3 g''(s). \tag{4}$$

Suppose now that the average velocity equals the velocity at the average time on each compact interval. Setting $g = f'$, we infer from (4) that $f'''(s) = 0$ and conclude the proof as above.

Here is a more elementary verification of the second case. By hypothesis,

$$(t-a)f'((a+t)/2) - f(t) + f(a) = 0. \tag{5}$$

If we differentiate both sides of (5) three times with respect to t, we obtain

$$(1/8)(t-a)f^{(4)}((a+t)/2) + (3/4)f^{(3)}((a+t)/2) - f^{(3)}(t) = 0. \tag{6}$$

Setting $t = a$ in (6), we conclude that $f^{(3)}(a) = 0$. Hence, f'' is constant.

Our final case is more difficult. If we assume only that the velocity at the average equals the average of the velocities, then we have

$$F((a+b)/2) = (1/2)(F(a) + F(b)) \tag{7}$$

where $F = f'$. The only continuous solutions to this equation, known as Jensen's equation, are linear functions; such solutions have constant derivatives, as desired. However, the existence of nonlinear solutions of (7) can be established with the aid of Zermelo's axiom of choice. So to complete our proof we must verify that, even though f' need not be *a priori* continuous, it must nevertheless be linear. This follows from two theorems of real analysis: f' is Lebesgue measurable (since it is a derivative [**1**, p. 288]); and, because it is convex, it must be continuous [**2**, p. 96]. Hence, as above, it must be linear.

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Odd Path Sums in an Edge-Labeled Tree

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According to one of many equivalent definitions, a tree is a graph on n nodes in which each of the $\binom{n}{2}$ pairs of nodes is connected by a unique path. Thus if each of the $n-1$ edges of a tree is labeled with an integer, then each of the $\binom{n}{2}$ pairs of nodes has associated with it a uniquely determined path sum. In the remarkable edge-labeled tree of John Leech (FIGURE 1) the integers have been chosen to

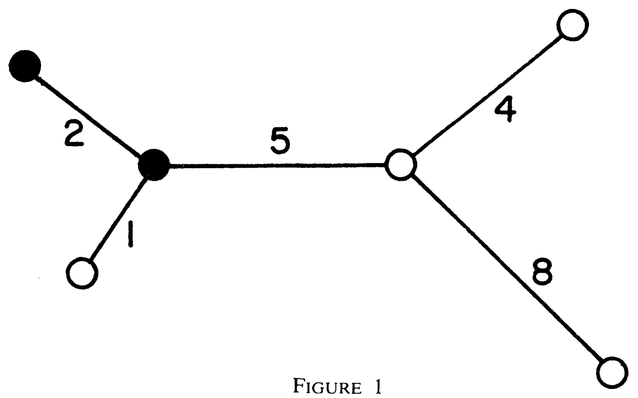

FIGURE 1

make the path sums run consecutively from 1 to $\binom{6}{2}$. It turns out that a key question concerns the number of odd path sums. A two-coloration trick, introduced below, will show that there are only a few possibilities for that number, over arbitrary labelings of the edges with integers.

Let an edge-labeled tree be given, and choose one node to start with black. Proceed along the edges to all the nodes of the tree, changing black to white or white to black across an odd edge, but keeping the same color across an even edge. When every node has been reached, each odd edge will connect a white node to a black node, and each even edge will connect black to black or white to white. The same property will hold for path sums as well. Indeed, the unique path between two nodes will have an odd sum if and only if it makes an odd number of color changes. Thus any path sum will be odd if and only if the path has a black node at one end and a white node at the other end. That is our two-coloration trick, and it answers our key question as follows: (BW) *If the edges of a tree with n nodes are labeled with integers, then the number of odd path sums must be equal to* $b \cdot w$, *where* $b + w = n$.

Evidence that (BW) is non-trivial appeared in **[1]** where John Leech asked for those trees on n nodes which could be edge-labeled to make the path sums take the consecutive values $1, \ldots, \binom{n}{2}$. He gave the complete answer for $n \leqq 6$, including the example of FIGURE 1, but said he could offer no information for $n > 6$. By virtue of (BW) we can easily show that it is impossible for certain values of n.

When $\binom{n}{2}$ is even, consecutiveness requires that exactly half of the path sums be odd. In this case we require nonnegative integers b and w such that $b + w = n$ and $2bw = n(n-1)/2$. This reduces quickly to the requirement that $n = (b-w)^2$. When $\binom{n}{2}$ is odd we must have $b + w = n$ and $2bw = n(n-1)/2 + 1$, which reduces quickly to $n = (b-w)^2 + 2$. Thus *an edge-labeled tree on n nodes cannot have the consecutive path sums* $1, \ldots, \binom{n}{2}$ *unless, for some integer m,* $n = m^2$ *or* $n = m^2 + 2$.

Asymptotically speaking, this says that for almost all values of n no tree on n nodes can be labeled as in FIGURE 1. Whether any at all exist for $n > 6$ is not known. I expect resolution of this question to be very difficult. I recommend that the reader compare these edge-labeling questions with the node-labeling results and questions to be found in the article **[2]** by Solomon W. Golomb and I thank him for several conversations and much encouragement.

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Subfields of Algebraically Closed Fields

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One of the most beautiful results in Galois theory is the theorem of Artin and Schreier which states that an algebraically closed field K has no subfields F such that the degree $[K:F]$ is finite and greater than 2. For example, the field C of complex numbers is algebraically closed, and hence has no subfields F with $[C:F]>2$ and finite. However, it is not difficult to show that C has uncountably many subfields F such that $[C:F]=2$, one of which, of course, is the field R of real numbers. The Artin-Schreier Theorem is often obtained (see [**1**, pp. 316–318], [**2**, pp. 66–67]) as a consequence of more general results and hence its simplicity is often obscured. It is the purpose of this paper to offer an elementary proof for fields of characteristic zero.

Let K be an algebraically closed field of characteristic zero, and F a subfield of finite codimension $[K:F]=n$. K is a normal extension of F (since all polynomials over F split in K) and the Galois group G of K over F has order n. Now if p is an odd prime dividing n, G has a subgroup of order p. Since K is normal over F, we can apply the fundamental theorem of Galois theory and conclude that there is an intermediate subfield of codimension p. On the other hand, if $n=2^r$, $r>1$, then G has order 2^r and thus contains a subgroup of order 4, giving rise to a subfield of codimension 4. Thus it suffices to show that K has no subfields of codimension 4 or an odd prime. (Our arguments will break down in the characteristic p case since we shall make use of primitive pth roots of 1 which do not then exist. The interested reader should consult [**2**] for a study of this case.)

Assume first that $[K:F]=p$ is a prime. Then G is cyclic of order p, say $G=\langle\sigma\rangle$, where $\sigma^p=1$. Since p is prime, there are no intermediate subfields strictly between K and F. That is, F has no nontrivial extensions of degree less than p. Since K is algebraically closed and of characteristic zero, it contains p distinct pth roots of 1. But any primitive pth root of 1 satisfies a polynomial over F of degree $p-1$ which would give rise to an intermediate field unless F itself contains all pth roots of 1. The nonexistence of intermediate subfields also implies that $K=F(\theta)$ for any $\theta\in K-F$.

We now use the above observations to show that there is an $\alpha\in K-F$ such that $\alpha^p\in F$. For any $\theta\in K-F$, set $\alpha=\theta+\zeta\sigma(\theta)+\zeta^2\sigma^2(\theta)+\cdots+\zeta^{p-1}\sigma^{p-1}(\theta)$, where $\zeta\in F$ is a primitive pth root of 1. Then $\sigma(\alpha)=\sigma(\theta)+\zeta\sigma^2(\theta)+\zeta^2\sigma^3(\theta)+\cdots+\zeta^{p-1}\sigma^p(\theta)$ which is equal to $\zeta^{-1}\alpha$. Thus α is not fixed by G, so $\alpha\notin F$ and thus $K=F(\alpha)$. However, $\sigma(\alpha^p)=\zeta^{-p}\alpha^p=\alpha^p$, so α^p is fixed by G, hence lies in F.

Starting with this α such that $K=F(\alpha)$ and $\alpha^p\in F$, we shall show by constructions leading to a contradiction that if $[K:F]=p$ is prime, then $p=2$. Since K is assumed to be algebraically closed, we let $\beta\in K$ be a fixed pth root of α. Thus $\beta^p=\alpha$, so $\beta^{(p^2)}=\alpha^p\in F$. Then $\beta^{(p^2)}=\sigma(\beta^{p^2})=[\sigma(\beta)]^{p^2}$, and $\sigma(\beta)=\mu\beta$, where μ is a p^2-root of 1. Observations about this element μ form the rest of the argument. If $\mu^p=1$, then $\sigma(\beta^p)=\mu^p\beta^p=\beta^p$, which implies that $\beta^p\in F$. But $\beta^p=\alpha\notin F$, so $\mu^p\neq 1$. Hence μ is a primitive p^2-root of 1, and μ^p is a primitive pth root of 1. As we observed earlier, this implies that $\mu^p\in F$. Therefore $\mu^p=\sigma(\mu^p)=[\sigma(\mu)]^p$, so $\sigma(\mu)=\mu\nu=\mu^{1+kp}$, where ν is a pth root of 1 and hence is of the form $(\mu^p)^k$ for some k. Since G is of order p, $\sigma^p(\beta)=\beta$. But since $\sigma(\beta)=\mu\beta$, by induction we also have

$$\sigma^p(\beta)=\mu^{1+(1+kp)+\cdots+(1+kp)^{p-1}}\beta.$$

We can now use the binomial theorem on this exponent and reduce each term modulo p^2 to account for the fact that $\mu^{(p^2)} = 1$. This eliminates all but the first two terms of each expansion, so we obtain

$$1 + (1 + kp) + \cdots + (1 + kp)^{p-1} \equiv p + kp(1 + 2 + \cdots + p - 1) \equiv p + \frac{kp^2(p-1)}{2} \pmod{p^2}.$$

Now if p is odd, this last expression is congruent to p, and we obtain $\sigma^p(\beta) = \beta = \mu^p\beta$, so $\mu^p = 1$, contradicting the previous conclusion that $\mu^p \neq 1$. Hence $p = 2$.

One final observation about the element μ will justify the conclusion that there is no subfield F of K with $[K : F] = 4$. We chose μ so that $\mu\beta = \sigma(\beta)$ where β was a pth root of α in K. Now if $\mu \in F$, $\sigma(\mu) = \mu$ and $\sigma^2(\beta) = \mu\sigma(\beta) = \mu^2\beta$ and by induction $\sigma^p(\beta) = \mu^p\beta$ which implies $\mu^p = 1$, again a contradiction. So $\mu \notin F$. In particular, if $[K : F] = 2$, then $i = \sqrt{-1} \notin F$ since i is a primitive fourth root of 1. But if $[K : F] = 4$, then G has order 4 and thus contains a subgroup of order 2. By the fundamental theorem of Galois theory, there exists an intermediate field L with $[K : L] = 2$. As we have seen, $\sqrt{-1} = i \notin L$, so clearly $i \notin F$. Thus $[K : F(i)] = 2$ but $F(i)$ contains i, a contradiction.

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An Ellipse Inequality

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A standard calculus problem asks for the maximum distance a normal line to a given ellipse can be from the center of the ellipse. In this note, we offer an elementary solution to this problem using only Cauchy's inequality, and then use this solution to derive a more general inequality.

The equation of the normal line to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (h, k) is given parametrically by $x = h + th/a^2$, $y = k + tk/b^2$. (These equations can be obtained without calculus by using the fact that the normal line bisects the angle formed by the lines joining the point (h, k) to the foci of the ellipse.) The distance r of the normal line from the center of the ellipse is obtained simply by requiring the normal line to be tangent to the circle $x^2 + y^2 = r^2$. What this means algebraically is that the circle and the normal line intersect in precisely one point. To discover where this occurs, we set the discriminant of the quadratic equation in t

$$x^2 + y^2 - r^2 \equiv t^2\left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right) + 2t + h^2 + k^2 - r^2$$

equal to zero:

$$r^2 = h^2 + k^2 - \left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)^{-1} = \frac{h^2k^2(a^2 - b^2)^2}{b^4h^2 + a^4k^2}$$

(In simplifying this expression, and in the subsequent analysis, we make frequent use of the relation $h^2/a^2+k^2/b^2=1$.) Now by Cauchy's inequality,

$$\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)(a^4k^2+b^4h^2)\geqq h^2k^2(a+b)^2$$

with equality if and only if $b^3h^2=a^3k^2$, or, equivalently, $h^2=a^3/(a+b)$ and $k^2=b^3/(a+b)$. It follows immediately that the maximum value of r^2 is $(a-b)^2$.

This reasoning can easily be extended to higher dimensions. For the n-dimensional ellipsoid $\Sigma x_i^2/a_i^2=1$ (the index on this and subsequent sums will run from 1 to n), the normal line through the point $(h_1,h_2,\ldots,h_n)$ is given parametrically by

$$x_i=h_i+th_i/a_i^2,\qquad i=1,2,\ldots n.$$

The distance r of this normal line from the center is found, as before, by requiring that the line be tangent to the sphere $\Sigma x_i^2=r^2$. Setting the discriminant of

$$t^2\Sigma h_i^2/a_i^4+2t+\Sigma h_i^2=r^2$$

equal to zero yields

$$r^2=\Sigma h_i^2-(\Sigma h_i^2/a_i^4)^{-1}. \tag{1}$$

To solve the n-dimensional problem of maximizing r^2 subject to $\Sigma(h_i/a_i)^2=1$, we reduce it to the previous 2-dimensional case by noting that the plane determined by the normal line and the center of the ellipsoid intersects the ellipsoid in an ellipse with the same center and the same normal line.

If we let a and b denote the semiaxes of this ellipse, our problem reduces to that of maximizing $(a-b)^2$ over all central sections of the ellipsoid. It is intuitively clear that this maximum should occur at a section formed by one of the coordinate planes and have as its value $\max_{i,j}(a_i-a_j)^2$. (This observation can be verified analytically by means of the obvious inequalities

$$\left(\min_i a_i\right)^{-2}\Sigma x_i^2\geqq\Sigma x_i^2/a_i^2\geqq\left(\max_i a_i\right)^{-2}\Sigma x_i^2.)$$

Thus the corresponding normal line lies in the plane (or planes) containing the maximum and minimum axes of the ellipsoid:

$$r_{\max}^2=\max_{i,j}(a_i-a_j)^2, \tag{2}$$

Suppose now that x_i and α_i are arbitrary real numbers (for $i=1,2,\ldots,n$), and let $h_i=x_i/R$ where $R^2=\Sigma(x_i/\alpha_i)^2$. Then $(h_1,h_2,\ldots,h_n)$ lies on the surface of the ellipsoid $\Sigma y_i^2/\alpha_i^2=1$, and thus, from (1) and (2),

$$\Sigma h_i^2-(\Sigma h_i^2/\alpha_i^4)^{-1}\leqq M$$

where $M=\max_{i,j}(\alpha_i-\alpha_j)^2$. By substituting x_i/R for h_i and simplifying, we obtain the right hand half in the double inequality

$$0\leqq\Sigma x_i^2\Sigma x_i^2/\alpha_i^4-\{\Sigma x_i^2/\alpha_i^2\}^2\leqq M\Sigma x_i^2/\alpha_i^2\Sigma x_i^2/\alpha_i^4. \tag{3}$$

The left hand inequality is just Cauchy's inequality. (The right hand inequality is, for this reason, called a complementary inequality.) Note that by setting $\Sigma x_i^2/\alpha_i^2=1$, we recapture our previous result (1) and (2). By letting $\alpha_i^2\to(a_i/b_i)$ and $x_i\to a_ix_i$, (3) is transformed into

$$\Sigma a_i^2x_i^2\Sigma b_i^2x_i^2-\{\Sigma a_ib_ix_i^2\}^2\leqq M'\Sigma a_ib_ix_i^2\Sigma b_i^2x_i^2, \tag{4}$$

where $M'=(\sqrt{M_2}-\sqrt{M_1})^2$ and $M_2=\max_i a_i/b_i$, $M_1=\min_i a_i/b_i$.

Finally, we show how (4) follows from

$$(5) \qquad (M_1+M_2)\Sigma a_ib_ix_i^2 \geqq M_1M_2\Sigma b_i^2x_i^2+\Sigma a_i^2x_i^2,$$

an inequality of Diaz and Metcalf [**1**, **2**] which is valid for all real numbers $b_i(\neq 0)$, a_i, x_i. Inequality (4) can be rewritten (after some elementary algebraic manipulation) as

$$(4') \qquad (M_1+M_2)\Sigma a_ib_ix_i^2\Sigma b_i^2x_i^2 \geqq \Sigma a_i^2x_i^2\Sigma b_i^2x_i^2 - \{\Sigma a_ib_ix_i^2\}^2 + 2\sqrt{M_1M_2}\Sigma a_ib_ix_i^2\Sigma b_i^2x_i^2.$$

That (5) is a stronger inequality than (4′) follows from the right hand side of (5) (multiplied through by $\Sigma b_i^2x_i^2$) being greater than or equal to the right hand side of (4′), since the corresponding difference is the perfect square $(\Sigma a_ib_ix_i^2 - \sqrt{M_1M_2}\Sigma b_i^2x_i^2)^2$.

References

[1] J. B. Diaz, F. T. Metcalf, Inequalities complementary to Cauchy's inequality for sums of real numbers, Inequalities, ed. O. Shisha, New York, 1967, pp. 73–77.

[2] D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970, pp. 61–62.

A Geometric Proof that $\sqrt{2}$ is Irrational

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The irrationality of $\sqrt{2}$ is a consequence of the following property of lattice squares, whose geometrical content is illustrated in FIGURE 1: *If a lattice square S contains a lattice square half as large, then S contains a smaller lattice square with the same property*. For suppose a lattice square of side m contains a lattice square half as large as side n. Then $m^2=2n^2$, so $(2n-m)^2=2(m-n)^2$ and thus $0<m-n<2n-m<m$. Therefore the lattice square of side $2n-m$ contains the lattice square of side $m-n$, equals twice that square, and is smaller than the given square. Repeated application of this construction would produce an endless sequence of smaller lattice squares, each of which is contained in the given lattice square and all of which are distinct. This is clearly impossible, however, for any lattice square of side m contains only a finite number of smaller lattice squares. Hence no lattice square can contain a lattice square half as large. This means that there are no integers m and n such that $m^2=2n^2$; in other words, $\sqrt{2}$ is irrational.

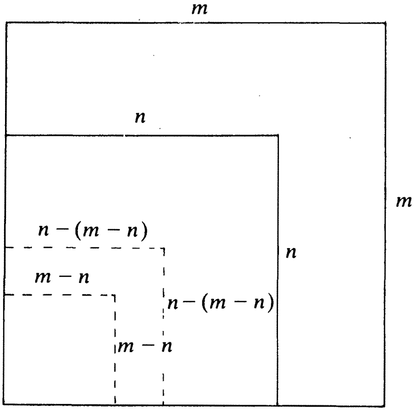

FIGURE 1

If $m^2=2n^2$, then $(2n-m)^2=2(m-n)^2$.

The same argument can be used to show that $\sqrt{3}$ is irrational.

Additive Sequences

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An additive sequence is a sequence A_i of integers such that $A_1 > 0$, $A_2 \geqq A_1$, and $A_{n+2} = A_n + A_{n+1}$. In other words, the first two terms are positive integers, the second term is greater than or equal to the first, and each successive term is the sum of the two preceding terms. The most famous such sequence is the Fibonacci sequence, in which $A_1 = A_2 = 1$. Another that is well known is the Lucas sequence, given by $A_1 = 1$, $A_2 = 3$. The Fibonacci sequence thus runs 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ..., and the Lucas sequence runs 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... A sequence such as 13, 21, 34, 55, ..., will be considered the same as the Fibonacci sequence, since by working backwards (computing $A_n = A_{n+2} - A_{n+1}$) one obtains as the smallest possible initial values $A_2 = 1$ and $A_1 = 1$. We shall assume that an additive sequence is characterized by its smallest possible initial values.

Among the first few terms of the Lucas sequence, only the first two, 1 and 3, also appear in the Fibonacci sequence. It is natural to wonder if there are any more such common terms. In this paper I shall prove that there are not, and shall establish a specific limit on terms common to *any* two additive sequences.

If A is an additive sequence, define $R_i = A_{i+1}/A_i$, $D_i = R_i - R_{i-1}$ (for $i \geqq 2$), and let $c(A) = A_1A_3 - A_2^2$. If F is the Fibonacci sequence, $c(F) = 1$. It has been known for years that $F_{i-1}F_{i+1} - F_i^2 = (-1)^i$ for all integers $i \geqq 2$. It can be shown easily by induction that, more generally, $A_{i-1}A_{i+1} - A_i^2 = (-1)^i c(A)$ for $i \geqq 2$. It follows that $D_i = (-1)^i c(A)/A_{i-1}A_i$, so $\Sigma_{i=2}^{\infty} D_i = -R_1 + \text{Lim}_{i\to\infty} R_i$ converges. The limit in this sum is well known for the Fibonacci sequence: it is the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$. The same result applies in general, since $R_{i+1}R_i = 1 + R_i$. Hence $\text{Lim}_{i\to\infty} R_i = \frac{1}{2}(1+\sqrt{5}) = \phi$, and $\Sigma_{i=2}^{\infty} D_i = \phi - R_1$, for any additive sequence.

Since $\Sigma_{i=2}^{\infty} D_i$ is an alternating series, the error in approximating its sum, using k terms, will be less in absolute value than the next term; hence $|(\phi - R_1) - (R_k - R_1)| < |D_{k+1}|$. A little rearrangement gives

(1) $$\left|\frac{A_{k+1}}{\phi} - A_k\right| < \frac{|c(A)|}{\phi A_{k+1}}$$

Now suppose N is a natural number satisfying

(2) $$|c(A)| < \phi N/2.$$

If $N = A_1$, for example, then either $A_2 = N$ or $A_2 > 2N$; in either case, $|c(A)| \geqq N^2 > \phi N/2$. So if N is in A, there is a natural number k such that $N = A_{k+1}$. Combining this with (1) and (2) then gives

(3) $$\left|\frac{A_{k+1}}{\phi} - A_k\right| < \frac{1}{2}.$$

This shows that A_k is the closest integer to A_{k+1}/ϕ. Thus, given $N = A_{k+1}$, we can find A_k; we can proceed to calculate $A_{k-1} = A_{k+1} - A_k$, etc., eventually reaching the smallest possible initial values. Thus A is determined by N and (2).

It follows that if a natural number N in A satisfies $N > 2|c(A)|/\phi$ (or, in other words, $N > (\sqrt{5}-1)|c(A)|$), then N is in no other additive sequence B with $N > (\sqrt{5}-1)|c(B)|$. Thus we have proved the Common Integer Theorem: *If A and B are two different additive sequences, with $|c(A)| \geqq |c(B)|$, then they have no common term $N > (\sqrt{5}-1)|c(A)|$.*

PROBLEMS

Dan Eustice, Editor
Leroy F. Meyers, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before June 1, 1978.

1025. Let $\{a_n\}$ be a sequence of positive real numbers with $\Sigma\, a_n = \infty$ and $\Sigma\, a_n^2 < \infty$. For a given $C > 0$, the sequence $\{m_i\}$ of positive integers is such that $\Sigma\, a_n > C$, the sum being over those n such that $m_i < n \leqq m_{i+1}$.

a. Prove that there is a sequence $\{p_i\}$ with $m_i < p_i \leqq m_{i+1}$, such that $\Sigma\, a_{p_i} < \infty$.

b. Show by an example that $\Sigma\, a_{p_i}$ need not converge for all such $\{p_i\}$. [*W. C. Waterhouse, The Pennsylvania State University.*]

1026. *A* decomposition of a positive integer n is an ordered tuple $(n_1, n_2, \ldots, n_k)$ of positive integers such that $\Sigma_{i=1}^{k} n_i = n$. Find the total number of decompositions of n which are palindromes. For example, for $n = 4$, there are four such, namely: (4), (2,2), (1,1,1,1), and (1,2,1). [*Michael Capobianco, St. John's University.*]

1027. Let $f(k)$ be a real-valued function on the nonnegative integers. Suppose that $f(0) = 0$ and that $f(k)$ is a convex function. That is, for $k \geqq 1$, $f(k)$ is less than the average of $f(k-1)$ and $f(k+1)$. For integers $k, 1 \leqq k \leqq n$, define

$$F_n(k) = f(k)q + f(r), \quad \text{for} \quad n = kq + r,\ 0 \leqq r < k.$$

Prove that, for fixed n, $F_n(k)$ is strictly increasing for $1 \leqq k \leqq n$. [*Daniel B. Shapiro, The Ohio State University.*]

1028. Let ABC be a triangle and P_1, P_2, and P_3 be arbitrary points in the plane of ABC. Let arbitrary lines perpendicular to AP_i, BP_i, and CP_i determine triangles $A_iB_iC_i$ for $i = 1,2,3$. Now, let A_0, B_0, and C_0 be the respective centroids of triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$. Show that the perpendiculars from A, B, and C on the sides of triangle $A_0B_0C_0$ concur. [*Leon Gerber, St. John's University.*]

ASSISTANT EDITORS: Don Bonar, *Denison University*; William A. McWorter, Jr., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgement of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q649. Show that every rational number r may be written as the sum of four or fewer rational cubes. [*Norman Schaumberger, Bronx Community College.*]

Q650. It is easy to see that if p is a prime, then $\binom{p}{k} \equiv 0 \pmod{p}$ for $k = 1, 2, \ldots, p-1$. Prove the generalization that for any positive integer n, $\binom{n}{k} \equiv 0 \pmod{n}$ if $(n, k) = 1$ and $k = 1, 2, \ldots, n-1$. [*Edward T. H. Wang, Wilfrid Laurier University.*]

Solutions

Number of Hexagons **March 1976**

975. In FIGURE 1, n, the length of the base, is 5 units, and $f(n)$, the number of different regular hexagons, is 6. Find a formula for $f(n)$. [*Charles L. Hamberg, Prairie View, Illinois, and Thomas M. Green, San Pablo, California.*]

Solution: Let t_n be the triangle with side n and note that t_k has $\binom{k+2}{2}$ vertices and that if the outer border of triangles is removed from t_n we obtain t_{n-3}. Now, each of the $\binom{n-1}{2}$ vertices of t_{n-3} is the center of a unique hexagon whose perimeter partially lies on the perimeter of t_n. Since these $\binom{n-1}{2}$ hexagons are the only ones in t_n which are not in t_{n-3}, we have

$$f(n) = \binom{n-1}{2} + f(n-3).$$

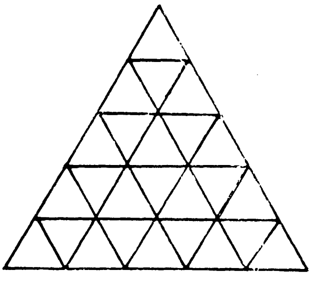

FIGURE 1

By recursion it follows that

$$f(n) = \sum_{j=1}^{[n/3]} \binom{n+2-3j}{2}$$
$$= ([n/3]/2)(3[n/3]^2 - 3n[n/3] + n^2 - 1).$$

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Also solved by J. Binz (Switzerland), Jordi Dou (Spain), Michael W. Ecker, Thomas E. Elsner, Kirk Fleming, Herta T. Freitag, Landy Godbold, William E. Gould, M. G. Greening (Australia), Eli Leon Isaacson, Jordan I. Levy, James McKim, Sidney Penner, F. G. Schmitt, Jr., G. W. Valk, Pambuccian Victor (Romania), James W. Walker, Edward T. Wang (Canada), Allan K. Wells, Western Maryland College Problems Seminar, Kenneth M. Wilke, and the proposers.

976. A road is to be built connecting two towns separated by a river whose banks are concentric circular arcs (see figure). If the road must bridge the river banks orthogonally, describe the minimum length road (assuming coplanarity). [*Miller Puckette and Steven Tschantz, 1975 U.S.A. International Mathematical Olympiad Team.*]

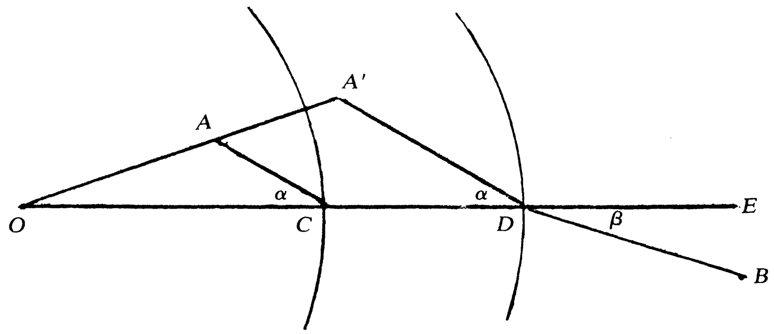

FIGURE 1

Solution (*adapted by the editors*). Suppose that the circular arcs, with center O, have radii r and R, and that the towns are at A and B, where $OA \leqq r \leqq R \leqq OB$. Expand the inner circle by a factor R/r, thus taking A to a point A' with $OA' \leqq R$. If $ACDB$ is an admissible road (orthogonal to the river banks), then $AC \| A'D$ by similar triangles. Minimizing $AC + CD + DB$, the length of the road, is equivalent to minimizing $(r/R)A'D + DB$, since CD is the constant $R - r$. The problem is then equivalent to that of finding a road $A'DB$ which can be traversed in minimum time, if the speed on the portion $A'D$ is R/r times that on the portion DB. But this problem is solved by Snell's law:

$$\frac{\sin\alpha}{\sin\beta} = \frac{R}{r},$$

where $\alpha = \angle ODA' = \angle OCA$ and $\beta = \angle BDE$.

An easy way to see that Snell's law determines the point D uniquely is to imagine looking into a crystal ball. A bubble inside the glass is seen as a single point, no matter where the observer is situated outside the ball.

MILLER PUCKETTE, Freshman
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STEVEN TSCHANTZ, Freshman
University of California

Also solved by Jordi Dou (Spain). Dou states that, in general, the point D is not constructible by straightedge and compass. Several incorrect solvers constructed a parallelogram CDBF and then stated that ACF must be a straight line. But since F depends on D, the problem is not equivalent to that of determining the shortest route between the points A and F.

Mr. P. and Ms. S

March 1976

977. Let x and y be two integers with $1 < x < y$ and $x + y \leqq 100$. Suppose Ms. S. is given the value of $x + y$ and Mr. P. is given the value of xy.

(1) Mr. P. says: "I don't know the values of x and y."

(2) Ms. S. replies: "I knew that you didn't know the values."

(3) Mr. P. responds: "Oh, then I do know the values of x and y."

(4) Ms. S. exclaims: "Oh, then so do I."

What are the values of x and y? [*David J. Sprows, Villanova University.*]

Editor's Comment. Proposal 977 is a succinct variation of some past problems in the Amer. Math. Monthly, especially E776, E1126, and E1156.

Solution. After (1) and (2) we know that x and y are not both prime numbers and that the sum s has no representation as sum of two primes.

$$M = \{11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53, 57, 59, 65, 67, 71, 77, 79, 83, 87, 89, 93, 95, 97\}$$

is the set of 24 possible values of s; exactly one of the numbers x and y is even, and the product p is even. 16 of the members of M have at least two distinct partitions of the form $2^m + q$, $m \geqq 2$, q an odd prime. If p is one of the corresponding products $2^m q$, then Mr. P. knows x and y, but not Ms. S., a contradiction to (4). We know now (but not Mr. P.) that only the sums $\{17, 29, 41, 53, 59, 65, 89, 97\}$ are possible. The partitions $29 = 2 + 27 = 13 + 16$, $41 = 4 + 37 = 16 + 25$, $53 = 16 + 37 = 32 + 21$, $59 = 16 + 43 = 32 + 27$, $65 = 4 + 61 = 32 + 33$, $89 = 16 + 73 = 64 + 25$ and $97 = 8 + 89 = 64 + 33$ show that Ms. S., after the positive exclamation (3) of Mr. P., is unable to determine x and y. Hence the unique possible sum is 17 with the partitions $17 = 2 + 15 = 3 + 14 = 4 + 13 = 5 + 12 = 6 + 11 = 7 + 10 = 8 + 9$. The factorizations $30 = 2 \cdot 15 = 5 \cdot 6$, $42 = 3 \cdot 14 = 2 \cdot 21$, $60 = 5 \cdot 12 = 3 \cdot 20$, $66 = 6 \cdot 11 = 2 \cdot 33$, $70 = 7 \cdot 10 = 2 \cdot 35$ and $72 = 8 \cdot 9 = 3 \cdot 24$ show that Mr. P. in these cases cannot give the answer (3). Hence $xy = 52$, and Mr. P. knows $x = 4$, $y = 13$. Ms. S., with 17 in her hand, concludes with the same arguments as we that $x + y = 4 + 13$. Therefore, the problem has the unique solution $x = 4$, $y = 13$.

PROBLEM SOLVING GROUP
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Also solved by K. V. Bhagwat & R. Subramanian (India), Thomas E. Elsner, Landy Godbold, Schuyler Grant, M. G. Greening (Australia), David Hammer, Eli Leon Isaacson, James McKim, William Nuesslein, Daniel O'Dell, Eric Rosenthal, St. Olaf Problems Group, Bill Stiefel, G. W. Valk, Pambuccian Victor (Romania), A. Bruce Wilson, and the proposer.

Positive Coefficients

May 1976

978. For $\lambda > 0$, let

$$(1 - x - y + axy)^{-\lambda} = \sum_{m,n=0}^{\infty} c_{m,n}^{(\lambda)} x^m y^n.$$

Show that $c_{m,n}^{(\lambda)} \geqq 0$ for all m, n if and only if $a \leqq 1$. [*L Carlitz, Duke University.*]

Solution. We have

$$S(x, y) = (1 - x - y + axy)^{-\lambda} = (1 - x)^{-\lambda} \left(1 - \frac{1 - ax}{1 - x}\, y\right)^{-\lambda}$$

$$= (1 - x)^{-\lambda} \sum_{n=0}^{\infty} \binom{n + \lambda - 1}{\lambda - 1} \left(\frac{1 - ax}{1 - x}\right)^n y^n.$$

The substitution $b = 1 - a$ gives

$$S(x, y) = \sum_{n=0}^{\infty} \binom{n+\lambda-1}{\lambda-1} \left[(1-x)^{-\lambda} \left(1 + \frac{bx}{1-x}\right)^n\right] y^n = \sum_{n=0}^{\infty} \binom{n+\lambda-1}{\lambda-1} \alpha_n^{(\lambda)}(x) y^n.$$

Since

$$\alpha_n^{(\lambda)}(x) = \sum_{j=0}^{\infty} \binom{n}{j} \frac{(bx)^j}{(1-x)^{\lambda+j}} = \sum_{j=0}^{\infty} \left[\binom{n}{j} b^j x^j \cdot \sum_{i=0}^{\infty} \binom{i+\lambda+j-1}{\lambda+j-1} x^i\right],$$

the coefficient of x^m is

$$d_{m,n}^{(\lambda)} = \sum_{j=0}^{m} \binom{n}{j} b^j \binom{\lambda+m-1}{\lambda+j-1},$$

and the coefficient of $x^m y^n$ in $S(x, y)$ is

$$c_{m,n}^{(\lambda)} = \binom{n+\lambda-1}{\lambda-1} d_{m,n}^{(\lambda)}.$$

If $a \leqq 1$, then $b \geqq 0$, and obviously $d_{m,n}^{(\lambda)} \geqq 0$ for all m, n. Therefore $a \leqq 1$ is sufficient for $c_{m,n}^{(\lambda)} \geqq 0$ for all m, n. To prove the only if part, let $a > 1$ or equivalently $b < 0$ and take $n = 1$, $m > -\lambda/b$. Now

$$d_{m,1}^{(\lambda)} = \binom{\lambda+m-1}{\lambda-1} + b\binom{\lambda+m-1}{\lambda} = \binom{\lambda+m-1}{\lambda-1}\left(1 + \frac{bm}{\lambda}\right) < 0,$$

which implies $c_{m,1}^{(\lambda)} < 0$. Hence $a \leqq 1$ is necessary.

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Also solved by M. T. Bird, Richard A. Groeneveld, J. M. Stark, and the proposer.

The Number of Permutations **May 1976**

979. Define $P(m, n)$ to be the number of permutations of the first n natural numbers for which m is the first number whose position is left unchanged. Clearly $P(1, n) = (n-1)!$ for all n. Show that, for $m = 1, 2, \ldots, n-1$,

$$P(m+1, n) = P(m, n) - P(m, n-1).$$

[*Mike Chamberlain and John Hawkins, University of Santa Clara.*]

Solution I: Define $\pi(m, n)$ to be the set of permutations of the first n natural numbers for which m is the first number whose position is left unchanged, with $(a_1, \ldots, a_n)$ an element of $\pi(m, n)$. Define $\pi(m, m+1, n)$ to be the subset of $\pi(m, n)$ for which $a_{m+1} = m+1$, and let $\pi'(m, m+1, n)$ be the complement of $\pi(m, m+1, n)$ in $\pi(m, n)$. Define f: $\pi(m+1, n) \leftrightarrow \pi'(m, m+1, n)$ by $f(a_1, \ldots, a_n) = (a_1', \ldots, a_n')$, where $a_{m+1}' = a_m \neq m+1$, $a_m' = a_p = m$, $a_p' = a_{m+1} = m+1$, and $a_k' = a_k$ if $k \neq m$, $m+1$, p.

Since f is one-to-one onto and $|\pi(m, m+1, n)| = |\pi(m, n-1)|$, we have

$$|\pi(m+1, n)| = |\pi'(m, m+1, n)| = |\pi(m, n)| - |\pi(m, m+1, n)| = |\pi(m, n)| - |\pi(m, n-1)|.$$

Therefore, $P(m+1, n) = P(m, n) - P(m, n-1)$.

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Livingston, New Jersey

Solution II: There exist $(n-1-i)!$ permutations keeping fixed both m and some given set of i out of the first $m-1$ natural numbers ($0 \leqq i \leqq m-1$). This set of i can be chosen in $\binom{m-1}{i}$ ways. Thus by inclusion and exlusion we get

$$P(m,n) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (n-1-i)!.$$

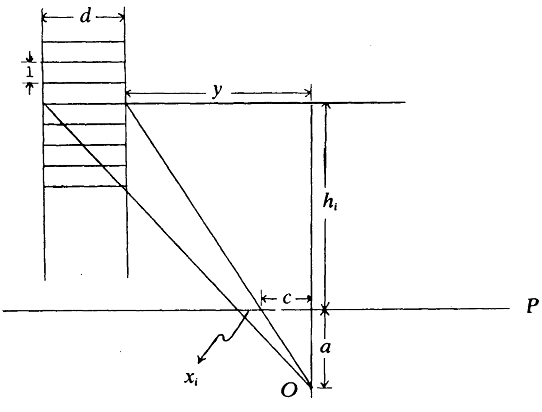
Hence the recursion follows immediately by using $\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$.

HEIKO HARBORTH
Braunschweig, West Germany

Also solved by J. C. Binz (Switzerland), Kirk Fleming, Richard A. Gibbs, Landy Godbold, M. G. Greening (Australia), Richard A. Groeneveld, Eli Leon Isaacson, Lael F. Kinch, Jordan I. Levy, Graham Lord (Canada), William Moser (Canada), Lawrence A. Ringenberg, David M. Rocke, Martin Schechter, Scott Smith, G. W. Valk, James W. Walker, Paul Y. H. Yiu (Hong Kong), and the proposers.

The Right Track **May 1976**

980. Show that in a perspective drawing of a straight railroad track which is at right angles to the image plane the reciprocals of the images of the ties form an arithmetic progression. [*Peter Ungar, New York University.*]



Solution: The diagram is a picture of the railroad tracks from above, where P is the image plane. The observation point O need not be in the plane of the tracks. However, perpendicular projection onto the plane of the tracks leaves x_i (the length of the image of the ith tie) unchanged. Thus, without loss of generality, O will be assumed to be in the plane of the tracks. We need to show that $1/x_{i+1} - 1/x_i$ is a constant independent of i. Using the diagram, we have $c/a = y(h_i + a)$ and $(x_i + c)/a = (d+y)/(h_i + a)$. Subtracting yields $x_i/a = d/(h_i + a)$ and $1/x_i = (h_i + a)/ad$. Therefore,

$$\frac{1}{x_{i+1}} - \frac{1}{x_i} = \frac{(h_{i+1} - h_i)}{ad} = \frac{1}{ad}.$$

ELI LEON ISAACSON
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Also solved by Thomas E. Elsner, Howard Eves, Richard A. Gibbs, Landy Godbold, Edward G. Hannon, Scott Smith, J. M. Stark, Paul Y. H. Yiu (Hong Kong), and the proposer.

An Arc of a Circle **May 1976**

981. Show that if a smooth curve in R^3 has the property that each principal normal line passes through a fixed point, then the curve must be an arc of a circle. [*Steven Jordan, University of Illinois at Chicago Circle.*]

Solution: Let $X(s)$, $a \leqq s \leqq b$, be the curve, parametrized by arc length, and let $T(s)$, $N(s)$, and $B(s)$ be, respectively, the unit tangent, normal, and binormal vectors to X at $X(s)$. By hypothesis, we know that there is a constant vector A and a function $\lambda(s)$ such that $X(s)+\lambda(s)N(s)=A$ for all s. Differentiating and applying the Frenet formulas, we find that $(1-\kappa(s)\lambda(s))T(s)+\lambda'(s)N(s)+\lambda(s)\tau(s)B(s)=0$, where $\kappa(s)$ is the curvature and $\tau(s)$ is the torsion. Since for each s, $T(s)$, $N(s)$, and $B(s)$ form an orthonormal basis for $\mathbb{R}^3$, we conclude that $\lambda'(s)=0$ for all s and so $\lambda(s)=\lambda$, a constant. We have

$$|X(s)-A|=[(X(s)-A)\cdot(X(s)-A)]^{1/2}=[\lambda N(s)\cdot\lambda N(s)]^{1/2}=|\lambda|.$$

Thus $X(s)$ is an arc of a circle of radius $|\lambda|$ and center A. We also have that $\kappa(s)=1/|\lambda|$ and $\tau(s)=0$.

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Also solved by Howard Eves, Donald C. Fuller, Edward G. Hannon, Paul T. Karch & George A. Novacky Jr., Lew Kowarski, Adam Riese, J. M. Stark, Paul Y. H. Yiu (Hong Kong), and the proposer.

Closure of $\{f(n+1)/f(n)\}$ **May 1976**

982.* Let $f(n)$ be the sum of all the positive divisors of the positive integer n, including 1 and n. Let A denote the set of all rational numbers of the form $f(n+1)/f(n)$. Determine the closure of A in the set of real numbers. [*Roy DeMeo, Jr., Franklin Square, New York.*]

Editors' comment. This problem is the same as Problem 6107* in the American Mathematical Monthly for August-Sepatember 1976. Bob Prielipp and Paul Erdös provided the references listed below to this and related problems. Erdös further provides the generalization that if $f(n)\geqq 0$ is a multiplicative number theoretic function with $f(p^\alpha)\to 1$ as $p^\alpha\to\infty$ and if $\Sigma_{p_k}|f(p_k^\alpha)-1|=\infty$, then $\{f(n+1)/f(n)\}$ is everywhere dense in $(0,\infty)$.

References

A. Schinzel, Generalization of a theorem of B. S. K. R. Somayjula on the Euler's function $\phi(n)$, Ganita (Lucknow), 5 (1954) 123–128.

A. Schinzel, On functions $\phi(n)$ and $\sigma(n)$, Bull. Acad. Pol. Sci. III, 3 (1955) 415–419.

P. Erdös, Some remarks on Euler's ϕ function, Acta Arith., 4 (1958) 10–19.

P. Erdös and A. Schinzel, Distributions of the values of some arithmetical functions, Acta Arith., 6 (1961) 473–485.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q649. We have $r=(r/6+1)^3+(r/6-1)^3+(-r/6)^3+(-r/6)^3$.

Q650. Since $\binom{n}{k}=(n/k)\binom{n-1}{k-1}$, we have $n\mid k\binom{n}{k}$. From $(n,k)=1$, we get $n\mid\binom{n}{k}$.

REVIEWS

Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Reviews of books are adapted from the Telegraphic Reviews in the American Mathematical Monthly.

Browder, Felix E., *Does pure mathematics have a relation to the sciences?,* American Scientist 64 (September-October 1976) 542-549.

"Pure mathematics is that part of mathematical activity that is done without explicit or immediate consideration of direct application to other intellectual domains...This new definition points to the distinction between the short-run applications of mathematics with forethought on the one hand, and the long-run and surprising application of mathematics, on the other." The author proceeds to examine numerous points on contact between sophisticated concepts of pure mathematics and interesting potential scientific applications.

Branch, Taylor, *New frontiers in American philosophy,* New York Times Magazine (14 August 1977) 12-22, 46-47, 62-67.

Extended portrait of the milieu and the man--Saul Kripke (Princeton), at age 36 touted as one of the two or three pre-eminent philosophers of the English-speaking world. Kripke, who has distinguished himself as a mathematician, logician, and philosopher, has concerned himself with such diverse topics as ordinals in set theory, modal logic, the nature of truth, and the philosophy of human emotion.

Thomsen, Dietrick E., *Physics without limits,* Science News 112 (17 September 1977) 186-187.

Popular account of the progress of Donald Greenspan (Wisconsin) in reformulating physical laws on the basis of discrete models instead of continuous ones. The successes so far--in Newtonian dynamics, fluid dynamics, continuum mechanics and special relativity--raise the question of just how essential calculus is to physics.

Appel, Kenneth and Haken, Wolfgang, *The solution of the four-color map problem,* Scientific American 237 (October 1977) 108-121, 152.

An extensive nontechnical report by the mathematicians who solved the Four Color conjecture, describing not only the logic of their proof, but also the evolution of their research. Adapted from the volume *Mathematics Today: Twelve Informal Essays* to be published in 1978 by Springer-Verlag and the Conference Board of the Mathematical Sciences.

Hess, Adrien L., Four-Dimensional Geometry, An Introduction, NCTM, 1977; iii + 28 pp, $1.60 (P).

The definition and history of four-dimensional geometry, along with drawings and models with accompanying instructions.

Rucker, Rudolf v.B., Geometry, Relativity and the Fourth Dimension, Dover, 1977; 133 pp, $2.75 (P).

An extraordinary exploration of the geometry of space-time, beginning with elementary concepts of four-dimensional geometry and concluding with speculation (based on the most current work in theoretical physics) concerning the geometric nature of fundamental physical reality.

Conway, Gordon R., *Mathematical models in applied ecology*, Nature 269 (22 September 1977) 291-297.

A review article which distinguished among large-scale strategic models, policy models, and tactical models and compares their relative merits in fish harvesting, malaria control, and pest control. No unified theory emerges, but a general conclusion is that the goal of the modelling is to design systems which are resilient to all conceivable ecological, economic, social and political change.

Kolata, Gina Bari, *Cryptography: on the brink of a revolution?*, Science 197 (19 August 1977) 747-748.

An NP-complete problem (the knapsack problem) leads to a technique of developing "trap-door one-way" enciphering functions which cannot be deciphered without special information. This differs from the present cryptographic situation where knowing the enciphering function is enough to determine the deciphering function.

Golomb, Michael, *Paul Erdös: addenda*, Science 196 (27 May 1977) 938.

Relates how Erdös's "occasional expressions of sympathy for and monetary contributions to various causes, his signing of petitions, his correspondence with foreign mathematicians, and other matters" led to harassment by the U.S. State Department, who effected the loss of his job at Notre Dame by refusing him permission to reenter the U.S. after the 1954 International Congress until 1963.

Rubinstein, Moshe F., Patterns of Problem Solving, Prentice-Hall, 1975; xvi + 544 pp, $15.95.

Developed from a campus-wide interdisciplinary course. Conceives of problem-solving as a dynamic process involving a variety of disciplines. Notable emphasis on attitudes and sharp focus on values in problem-solving. Instructor's manual and solutions manual available.

Swetz, Frank J. and Kao, T.I., Was Pythagoras Chinese? An Examination of Right Triangle Theory in Ancient China, Penn St U Pr and NCTM, 1977; 75 pp, $4.40 (P).

Timely offering of evidence for ancient Chinese mathematical accomplishments. The book is devoted to a translation of a chapter of *Chiu chang suan shu* which contains 24 problems on right triangles (some of which later found their way into Sanskrit treatises). The larger work, which dates from the third century B.C., also contains examples of the use of negative numbers, "Horner's" method, row operations on matrices, as well as the formula for the volume of a truncated triangular right prism. The reader must admit the richness of the Chinese tradition, not in deductive exposition of properties of right triangles, but in geometric insight and computational proficiency. (The catchy title is a take-off on Hans Breuer's *Columbus Was Chinese: Discoveries and Inventions of the Far East*, Herder, 1972.)

Dembart, Lee, *Theory of groups: a key to the mysteries of math*, New York Times (17 May 1977) 35, 55.

Feature article focussing on the 32-step program outlined by Daniel Gorenstein (Rutgers) for determining all finite simple groups--a program which he predicts will be accomplished within four or five years.

Straffin, Philip D., Jr., *Homogeneity, independence, and power indices,* Public Choice 30 (Summer 1977) 107-118.

A comparison, using probability models, of the two most widely used measures of voting power, the power indices of Shapley-Shubik and Banzhaf. The author shows that they can give widely different results and suggests a heuristic criterion for which of them is appropriate to a given voting situation. Chief illustration: the proposed process for amending the Canadian Constitution.

Stigler, Stephen M., *Eight centuries of sampling inspection: the Trial of the Pyx,* J. American Statistical Association 72 (1977) 493-500.

Delightful account of a sampling inspection scheme in operation at the Royal Mint at London for eight centuries. The author notes a defect in the manner in which the critical levels were set and examines the possibility that Newton while Master of the Mint may have grasped the nature of the defect.

Senechal, Marjorie and Fleck, George (ed), Patterns of Symmetry, U of Mass Pr, 1977; viii + 152 pp, $12.

An inspiring collection of essays for the general reader, beautifully printed. Several essays are from papers presented at the Vanderbilt Symmetry Festival at Smith College in 1973. Topics include color symmetry; change ringing (in which it is revealed precisely how the mechanics of ringing the bells determines the art form and hence the mathematics); and symmetry in typographic ornament, molecules, crystals, plants and literature.

Rabin, Michael O., *Complexity of computations,* Comm. Assoc. Comp. Mach. 20 (1977) 625-633.

After listing typical computational tasks, this article discusses the main questions that are asked about them and introduces the concepts that arise from the questions. An excellent short summary of the state of the art follows, and the author suggests further research on secure communications and very large data bases. An expanded version of 1976 Turing lecture.

The Demand-Revealing Process, Public Choice 29:2 (Spring 1977) Special Supplement.

The demand-revealing process is a new way to make collective decisions which is attracting the interest of mathematicians, political scientists, and economists, all of whom are represented here. In a demand-revealing process, participants are given the choice of accepting the decision which would be made without their participation, or of changing the decision to whatever they want, upon payment of an amount of money equal to the net cost of the change to all other persons.

Calinger, Ronald, Gottfried Wilhelm Leibniz, Edwin B. Allen Math. Mem., Rensselaer Poly. Inst., 1976; ix + 102 pp, $5 (P).

Concise vignette of his life, with brief discussions of his philosophy and his posthumous reputation. Calculus students will enjoy Leibniz's own account of how he came to study mathematics seriously (p. 12).

Woods, G.T., *Chemically, the same or different?,* Mathematical Gazette 60 (December 1976) 247-256.

Exploration of common ground between network theory in mathematics and isomerism in chemistry, leading to interpretation of the diagrams drawn by the chemist into the language of geometry and topology.

Gardner, Martin, The Incredible Dr. Matrix, Scribners, 1976; 256 pp, $8.95.

All of the 18 columns through 1975 that Gardner has written for *Scientific American* on the whimsical Dr. Matrix, so that this book supplants *The Numerology of Dr. Matrix* (Simon & Schuster, 1967).

Benson, William H. and Jacoby, Oswald, The New Recreations with Magic Squares, Dover, 1976; viii + 198 pp, $4 (P).

Classification, methods of construction (many new--e.g., a cyclical method applicable to both even and odd squares), problems of enumeration, and proofs (good examples of modular arithmetic). Presumes only high school algebra.

Richardson, Jane S., *β-sheet topology and the relatedness of proteins*, Nature 268 (11 August 1977) 495-500.

An example of the usefulness of the topology of connectedness in the study of the organization of proteins.

Jennings, A., *Matrices, ancient and modern*, Bulletin of the Institute of Mathematics and Its Applications (13 May 1977) 117-123.

A good survey of linear computation, giving some history of matrices but focussing on snags in matrix computation and six modern (Post 1950) developments in that art.

Linn, Charles F. (ed), The Ages of Mathematics, V. I-IV, Doubleday, 1977; $5.95 each. V. I: The Origins, Michael Moffatt, 137 pp; V. II: Mathematics East and West, Charles F. Linn, 151 pp; V. III: Western Mathematics Comes of Age, Cynthia Conwell Cook, 151 pp; V. IV: The Modern Ages, Peter D. Cook, 137 pp.

A flawed but valuable "non-mathematical survey" of the history of mathematics, designed for high-school students. (For a longer review, see *Amer. Math. Monthly* 84 (August-September 1977) 584.)

Fuchs, H., Kedem, Z.M. and Useltom, S.P., *Optimal surface reconstruction from planar contours*, Comm. Assoc. Comp. Mach. 20 (1977) 693-702.

Applications of directed graphs to reconstruction of a three-dimensional object from cross-sections of parallel planes.

Austing, Richard H., *The GRE advanced test in computer science*, Comm. Assoc. Comp. Mach. (20 September 1977) 642-645.

Description of the test which was recently introduced into the Graduate Record Examination Program. Content specifications (with rationale) are given, and a set of sample questions is included.

Goldstein, A.A., *Optimal temperament*, SIAM Review 19 (July 1977) 554-562.

Shows that musical scales can be constructed by finding solutions of inconsistent systems of linear equations with various criteria of optimality. Several historical scales are generated as examples.

Sussman, Hector J., *Letters: Catastrophe theory: a skeptic*, Science 197 (26 August 1977) 820-821.

Further correspondence, following G.B. Kolata's article on catastrophe theory (15 April 1977, 287, 350-351) and letters from readers (17 June 1977, 1270-1272).

NEWS & LETTERS

MATHEMATICS MODULES

In the winter and spring of the 1976-1977 academic year a survey of available modular materials in undergraduate mathematics was conducted by The Modules and Monographs in Undergraduate Mathematics and Its Applications Project (UMAP) with the help of the MAA Special Projects Office. The result is a collection of 424 module description sheets bound and indexed in the *Index and Description of Available Mathematical Modules:* Volume 1 (June 1977). The module description sheets give information about content, field, prerequisites, post-options, medium, availability, cost, and more.

The *Index* is now available at a cost of $2.50 to cover printing, handling, and mailing. You may order your copy from EDC/UMAP; 55 Chapel Street, Newton, Massachusetts 02160. Checks should be made out to Education Development Center.

OTHER SOURCES

The main result of "The Number of Square Matrices of a Fixed Rank" by L. Verner (this *Magazine*, March 1977, pp. 95-96) is well-known. This and other related results appear in "Matrices Over a Finite Field" by S.D. Fisher and M.N. Alexander (*Amer. Math. Monthly*, 73 (1968) 639-641), and in "On the Foundation of Combinatorial Theory IV: Finite Vector Spaces of Eulerian Generative Functions" by J. Goldman and G.C. Rota (*Studies in Applied Math.*, XLX (1970) 239-258).

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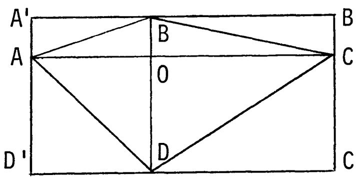
SHORT CIRCUIT

With regard to "Diamond Inequalities" (this *Magazine*, March 1977, pp. 96-98), here is the "more elementary proof" Klamkin was looking for. Let the segments AC and BD intersect orthogonally at 0. Then $AB + BC + CD + DA \geq 2(AC^2 + BD^2)^{1/2}$ as follows:



Let $A'B'C'D'$ be the rectangle circumscribed to $ABCD$ with sides parallel to AC and BD. Using the facts about the diagonals of a rectangle and the triangle inequality we easily get

$$AB + CD + BC + DA = (OA'+OC') + (OB'+OD')$$
$$\geq A'C' + B'D' = 2(AC^2+BD^2)^{1/2}$$

with equality if and only if 0 lies on both diagonals of $A'B'C'D'$, i.e., iff $ABCD$ is a rhombus.

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A CURIOUS SEQUENCE

Kahan's curious sequence (this *Magazine*, November 1975, pp. 290-291; March 1976, p. 102 (letter)) may be found in problem E1413 from the *Amer. Math. Monthly* (67 (1960) 378) whose solution appears in the December issue (67 (1960) 1030-1031).

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LIMITS TO GROWTH?

David Smith's interesting article "Human Population Growth..." (this *Magazine*, September 1977, pp. 186-197) reminded me of a "doomsday" calculation I did after reading about the 1923 German inflation in Constance Reid's *Courant* (Springer-Verlag 1976). Reid's data (p. 97) can be tabulated as follows:

Date	t (days)	x (marks/$)
July 1	0	1.60×10^5
Oct. 1	92	2.42×10^8
Nov. 20	142	4.20×10^{12}

With only three data points there is an exact fit of the form

$$x(t) = x(0)(1-t/t_\infty)^{-1/c} ,$$

the solution of $dx/dt \propto x^{1+c}$. The values of the parameters work out to $c = 0.107375$, $t_\infty = 168.992$ days, giving 17 December 1923 as the day when the mark's value should have shrunk to zero. However, the introduction of the *Rentenmark* in October makes this prediction moot.

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COINING THE PROBLEM

"Counterfeit Coin Problems" (this *Magazine*, March 1977, pp. 90-92) claims to establish E.D. Schell as the originator of this category of problem. It is unlikely that Mr. Schell was the progenitor of this variety of problem.

Indeed, although Manvel claims that "The classic works of Loyd, Ball, Dudeney, and Kraitchik contain no such problem," Kraitchik's work does contain such a problem, tucked away in the back, and not listed in the table of contents. My issue of *Mathematical Recreations* by Maurice Kraitchik was printed in 1953 by Dover Publication, Inc. However, it is a reprint of the 1942 W.W. Norton & Company, Inc. edition. On pages 324 and 325 of my Dover edition is the twelve-coin problem and a solution.

A rather thorough discussion by T.H. O'Beirne in *Puzzles and Paradoxes* (pp. 214-215) attempts to trace the history of the problem, and O'Beirne arrives at the same conclusion as does Manvel, so the location of the problem in Kraitchik's classic work does indeed make one wonder just how classic Kraitchik's work is! It is also true that Schaaf's rather exhaustive *A Bibliography of Recreational Mathematics* does not have this problem referenced prior to 1945.

If indeed Schell is the originator of this problem type, Kraitchik's inclusion of the problem must have been new to the Dover edition--but such a fact is not indicated in the Dover edition.

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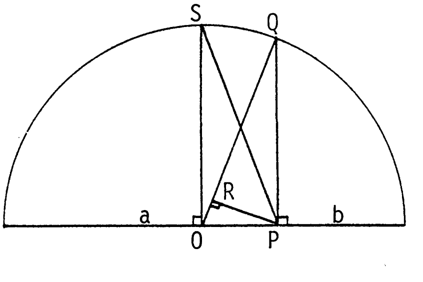
A MEANINGFUL FIGURE

In his figure "Proof Without Words: A Truly Geometric Inequality," Charles Gallant gives a geometric proof of the arithmetic-geometric means inequality (this *Magazine*, March 1977, p. 98). Albert Schild, in his article "Geometry of the Means" (*The Mathematics Teacher*, 67 (1974) 262-263), includes in the figure a segment whose length is the harmonic mean. The figure below shows the root-mean-square PS, the arithmetic mean $OS = OQ$, the geometric mean PQ, and the harmonic mean QR of the two positive lengths a and b. Clearly $PS > OS = OQ > PQ > QR$ when $a \neq b$.



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The sixth U.S.A. Mathematical Olympiad took place on May 3, 1977, and the problems were published that same month in this column. The following sketches of solutions were adapted by Loren Larson from Samuel Greitzer's pamphlet "Mathematical Contests for 1977."

1. Determine all pairs of positive integers (m,n) such that $(1+x^n+x^{2n}+\ldots+x^{mn})$ is divisible by $(1+x+x^2+\ldots+x^m)$.

Sol. The problem is identical to finding (m,n) such that

$$\frac{(x^{(m+1)n}-1)}{(x^{m+1}-1)}\cdot\frac{(x-1)}{(x^n-1)}$$

is a polynomial. For all positive integers m and n, $x^{m+1}-1$ as well as x^n-1 divide $x^{(m+1)n}-1$. But since the factors of $x^{(m+1)n}-1$ are distinct it is necessary that $x^{m+1}-1$ and x^n-1 have no common factors other than $x-1$. Thus, it is necessary and sufficient that $m+1$ and n be relatively prime.

2. ABC and $A'B'C'$ are two triangles in the same plane such that the lines AA', BB', CC' are mutually parallel. If ΔABC denotes the area of triangle ABC with an appropriate $\pm$ sign, etc, prove that

$$3(\Delta ABC + \Delta A'B'C') = \Delta AB'C' + \Delta BC'A' + \Delta CA'B' + \Delta A'BC + \Delta B'CA + \Delta C'AB.$$

Sol. We may assume the vertices are coordinatized as $A(a,k)$, $B(b,h)$, $C(c,0)$, $A'(a',k)$, $B'(b',h)$ and $C'(c',0)$. The result follows by direct substitution using the following facts:

(i) If a triangle in a plane has vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3), then its area is given by

$$\frac{1}{2}\det\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}$$

ii) If u, v, v' and w are column vectors with three components, then $\det(u,v,w)+\det(u,v',w)=\det(u,v+v',w)$.

3. If a and b are two of the roots of $x^4+x^3-1=0$, prove that ab is a root of $x^6+x^4+x^3-x^2-1=0$.

Sol. Let the roots of x^4+x^3-1 be denoted by a, b, c and d. Then we may write

(1) $(c+d) = -1 - a - b$

(2) $ab + a(c+d) + b(c+d) + cd = 0$

(3) $ab(c+d) + (a+b)cd = 0$

(4) $cd = -1/ab$.

Using (1) and (4) to eliminate $c+d$ and cd, (2) and (3) may be written as

(5) $r - (a+b) - (a+b)^2 - (1/r) = 0$

(6) $-r - r(a+b) - (a+b)(1/r) = 0$

where $r = ab$. Solving for $a+b$ in (6) and substituting into (5) gives $r^6+r^4+r^3-r^2-1=0$.

4. Prove that if the opposite sides of a skew (non-planar) quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and, conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

Sol. Denote the vectors from the origin to the vertices and the midpoints of the diagonals by A, B, C, D and P, Q respectively. We are given that

(1) $(A-B)\cdot(A-B) = (C-D)\cdot(C-D)$
$(B-C)\cdot(B-C) = (A-D)\cdot(A-D)$

and we want to prove

(2) $[(B+D)-(A+C)]\cdot(A-C) = 0$
$[(B+D)-(A+C)]\cdot(B-D) = 0$

where $\cdot$ denotes the scalar product. Adding and subtracting equations (1) yield equations (2), and conversely, adding and subtracting equations (2) give equations (1).

5. If a, b, c, d, e are positive numbers bounded by p and q, i.e., $0 < p \le a,b,c,d,e \le q$, prove that $(a+b+c+d+e)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}) \le 25 + 6(\sqrt{p/q}-\sqrt{q/p})^2$ and determine when there is equality.

Sol. Supposing a, b, c, d are given, we wish to find e which maximizes

$$(u+e)(v+\frac{1}{e}) = uv + 1 + ev + \frac{u}{e}$$

where

$$u = a + b + c + d, \quad v = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

Now $ev + (u/e) = (\sqrt{ev}-\sqrt{u/e})^2 + 2\sqrt{uv}$ which we see is a minimum when $\sqrt{ev} = \sqrt{u/e}$, or equivalently, when $e = \sqrt{u/v}$ (where $p \le \sqrt{u/v} \le q$) and takes on its maximum value when e is either p or q. In a similar fashion,

$$(a + b + c + d + e)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e})$$

takes on its maximum value when a, b, c, d, e take on the extreme values p or q. So suppose there are k p's and $5-k$ q's; we wish to maximize

$$(kp + (5-k)q)(\frac{k}{p} + \frac{5-k}{q})$$

which equals

$$k^2 + (5-k)^2 + k(5-k)(\frac{p}{q} + \frac{q}{p}).$$

Since

$$(\frac{p}{q} + \frac{q}{p}) = (\sqrt{p/q} - \sqrt{q/p})^2 + 2 ,$$

we must maximize

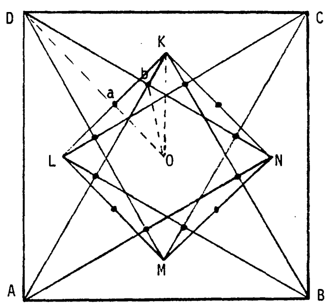
$$k(5-k)(\sqrt{p/q} - \sqrt{q/p})^2 + 25 .$$

This is a maximum when $k = 2$ or 3 which completes the problem: equality occurs when either two or three of the terms are equal to p and the rest equal to q.

SOLUTIONS TO THE 1977 INTERNATIONAL MATHEMATICAL OLYMPIAD

The September issue of Mathematics Magazine contained the problems from the 19th International Mathematical Olympiad, which took place in Yugoslavia in July 1977. Here are sketches of solutions to these olympian problems for readers who wish aid or confirmation.

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon. *(Holland)*



Sol. We will prove that $Oa = Ob$ and that $2\angle aOb = \angle bOK = 15°$. The result will follow from the symmetry of the figure by taking suitable reflections about OD, OL, and KM.

The segment AN is part of the perpendicular bisector of BK, from which it follows that $KN = NB$. By symmetry it follows that MBN is an equilateral triangle, say of side length s, and that $\angle CBN = 15°$. In ΔDBN, Ob joins the midpoints of DB and DN, so Ob is parallel to BN and equal to half of it. It follows that $Ob = s/2$ and $\angle bOK = 15°$. From this it follows that $Oa = \frac{1}{2}KN = \frac{1}{2}NB = \frac{1}{2}s$ and $\angle aOb = \angle DOK - \angle KOb = 30°$.

2. In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence. *(Viet Nam)*

Sol. There is no such sequence with seventeen terms, for suppose $(a_1, a_2, \ldots, a_{17})$ is such a sequence. Set A equal to the sum of the eleven possible sums of seven successive terms,

$$A = (a_1+\ldots+a_7)+(a_2+\ldots+a_8)+\ldots(a_{11}+\ldots+a_{17}).$$

By grouping together the first terms in each of these groups, and then the second terms, and so on, we can rewrite A as a sum of seven sums of eleven successive terms,

$$A = (a_1+\ldots+a_{11})+(a_2+\ldots+a_{12})+\ldots(a_7+\ldots+a_{17}).$$

From the first expression A is negative, but from the second, A is positive. Thus, no such sequence can be formed with seventeen or more terms.

Such a sequence does exist for sixteen terms--e.g., (5,5,-13,5,5,5,-13,5,5,-13,5,5,5,-13,5,5,).

3. Let n be a given integer > 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \ldots$. A number $m \in V_n$

is called indecomposable in V_n if there do not exist numbers p, $q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in the order of the elements of V_n will be considered the same.) *(Holland)*

Sol. By Dirichlet's Theorem, the arithmetical sequence $n - 1$, $2n - 1$, $3n - 1$, ...contains an infinite number of primes. Pick two of them, p and q. Since $p \equiv q \equiv -1 \pmod{n}$, neither p nor q are in V_n; however, p^2, q^2, and pq are in V_n since $p^2 \equiv q^2 \equiv pq \equiv 1 \pmod{n}$. Then $r \equiv p^2q^2 = (pq)(pq)$ is the required number, since p^2, q^2, and pq are indecomposable in V_n (their only factors are p and q).

(A proof not based on Dirichlet's Theorem can be given by taking the first two terms of the sequence above, namely $a = n-1$ and $b = 2n-1$, and setting $r = (a^2)(b^2) = (ab)(ab)$. It can then be shown that a^2, b^2 and (ab) are indecomposable in V_n).

4. a, b, A, B are given constant real numbers and $f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$. Prove that if $f(\theta) \geq 0$ for all real θ, then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$. *(Gt. Britain)*

Sol. Rewrite $f(\theta)$ in the form

$$f(\theta) = 1 - r\cos(\theta-\alpha) - R\cos 2(\theta-\beta)$$

where $r^2 = a^2 + b^2$, $R^2 = A^2 + B^2$, $\cos\alpha = a/r$, $\sin\alpha = b/r$, $\cos 2\beta = A/R$, $\sin 2\beta = B/R$. It now can be checked that if $r > \sqrt{2}$ then either $f(\alpha+45°)$ or $f(\alpha-45°)$ is negative, contradicting the hypothesis. Therefore, $r^2 = a^2 + b^2 \leq 2$. In the same way, if $R > 1$ then either $f(\beta)$ or $f(\beta+90°)$ is negative, counter to the hypothesis. Therefore $R^2 = A^2 + B^2 \leq 1$.

5. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a,b), given that $q^2 + r = 1977$. *(W. Germany)*

Sol. We have

(1) $a^2 + b^2 = q(a+b) + r$

(2) $1977 = q^2 + r$.

From (2), $q^2 \leq 1977$, so $q \leq 44$. Also,

$$q + 1 > \frac{a^2+b^2}{a+b} = (a+b) - \frac{2ab}{a+b},$$

and therefore

$$45 > (a+b) - \frac{2ab}{a+b} \geq (a+b) - \frac{a+b}{2}.$$

From this we conclude that $a + b \leq 90$ and $r \leq 89$. Therefore, from (2), $1977 \geq q^2 \geq 1888$, from which we can conclude that $q = 44$ and $r = 41$. Rewriting (1) using these values, we get $(a-22)^2 + (b-22)^2 = 1009$. The only solution to this equation is $\{|a-22|, |b-22|\} = \{15,28\}$ and therefore the only possible pairs are (50,37), (37,50), (50,7) and (7,50).

6. Let $f(n)$ be a function defined on the set of all positive integers and taking on all its values in the same set. Prove that if $f(n+1) > f(f(n))$ for each positive integer n, then $f(n) = n$ for each n. *(Bulgaria)*

Sol. We will prove, by induction, that for every positive integer n, $f(k) = n$ iff $n = k$.

There is some integer k such that $f(k) = 1$, for if not

$$f(k+1) > f(f(k)) > f(f(f(k)-1)) > \ldots$$

is an infinitely decreasing sequence of positive integers, an obvious contradiction. If $f(k) = 1$ for $k > 1$, then $1 = f(k) > f(f(k-1)) \geq 1$, also a contradiction. Therefore, the desired proposition holds for $n = 1$.

Suppose that we have proved the proposition for all integers less than n. As before, we can prove that $f(k) = n$ for some integer k, since if not we can construct an infinitely decreasing sequence of positive integers (start with any $k > n$). So suppose that $f(k) = n$ and that $k > n$. Then, since $f(f(k-1)) < f(k)$, we have $f(f(k-1)) \leq n - 1$. Applying the inductive principle twice, this last inequality holds iff $k - 1 \leq n - 1$, or equivalently, iff $k \leq n$, a contradiction. Therefore $f(n) = n$, and by induction, the proof is complete.

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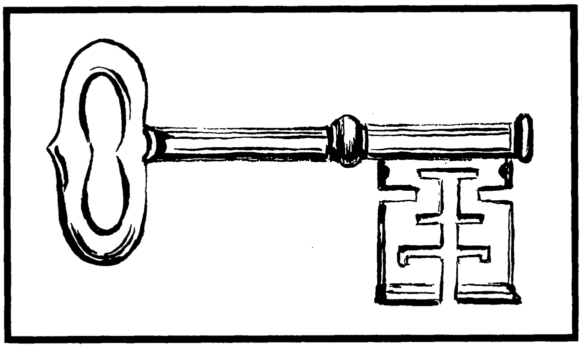

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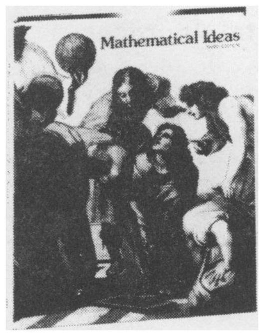
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